Set Systems of Finite Character and Equivalents of Boolean Prime Ideal Theorem

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Contents

1 Introduction

Sets are the fundamental objects upon which modern mathematics is built. In particular, the axioms of Zermelo-Fraenkel (ZF), among many other set theories developed in the last century, are those most widely accepted as foundation for mathematics. One often adds the Axiom of Choice to ZF and obtains the stronger set theory ZFC. By the works of Kurt Gödel and Paul Cohen, AC is proved to be independent of ZF, the other axioms of ZFC. It is well known that there are many principles equivalent (in ZF) to the Axiom of Choice. The most famous ones among them are the Well-ordering Theorem and Zorn's Lemma. These principles are characterized by their *non-constructiveness*; they assert the existence of a certain object, but do not provide a way to construct it. A typical kind of such principles asserts that a maximal element exists in some ordered set. Zorn's Lemma is an example:

Zorn's Lemma. *Let $(P, \leq)$ be a poset such that every chain of P has an upper bound. Then P has a maximal element, i.e. an element $m \in P$ such that if $x \in P$ with $m \leq x$ then $x = m$.*

In this paper we are especially interested in non-constructive principles for *systems of finite character*.

Definition 1.1. If X is a set, a collection $\mathcal{A}$ of subsets of X is called a *system* on X. Moreover, $\mathcal{A}$ has *finite character* whenever for every $A \in \mathcal{A}$, $A \in \mathcal{A}$ if and only if every finite subset of A is in $\mathcal{A}$.

One of such principles is the *Tukey-Teichmüller Theorem*; we will refer to it as TT. It ensures the existence of a maximal element, w.r.t. the inclusion, in systems of finite character.

Tukey-Teichmüller Theorem. *Let X be a set and $\mathcal{A}$ be a non-empty system of finite character on X. Then $\mathcal{A}$ has a maximal element.*

Also the Tukey-Teichmüller Theorem is known to be equivalent to the Axiom of Choice. In Section 2, we will discuss further about TT and give a proof of its equivalence with the Zorn's Lemma.

In the thrid Section, we will turn to the discussion about Boolean algebra, the Boolean Prime Ideal Theorem (PIT) and the Ultrafilter Lemma (UL). PIT is also a non-constructive principle, but is strictly weaker than the Axiom of Choice in ZF. In this paper, we will encounter many non-constructive principles that are equivalent to this Prime Ideal Theorem.

Then, in Section 4, we will again treat a principle in systems of finite character. As the first one of the two restrictions of the Tukey-Teichmüller Theorem, the *Restricted Tukey-Teichmüller Theorem* will be introduced. This restriction will turn out to be equivalent to the Prime Ideal Theorem. Also many application of this principle will be presented.

In the next, fifth Section, another restriction of the TT, named Finite Cutset Lemma, will be the central principle to study. Also this restriction will turn out to be equivalent to PIT, by a direct equivalence proof with the other restriction RTT. Via this principle, the equivalences between PIT and several other principles, including Alexander's Subbase Theorem from Topology, will be established too. At the end of Section 5, we will have stated a full proof of the following equivalences.

Theorem 1.2. *These principles are equivalent in ZF:*

1. *Boolean Prime Ideal Theorem (PIT)*
2. *Ultrafilter Lemma (UL)*
3. *Restricted Tukey-Teichmüller Theorem (RTT)*
4. *Finite Cutset Lemma (FC)*
5. *Intersection Lemma (IL)*
6. *Alexander's Subbase Theorem (AS)*
7. *Cowen-Engeler Lemma (CE)*
8. *Generalized Consistency Theorem (GC)*

In the last, 6th Section, we will observe some applications in Logic. We will see that the Restricted Tukey-Teichmüller Theorem, which is developed in Section 4, naturally proves important results for the propositional logic.

1.1 Notations

Here we introduce some notations, with which one may or may not be familiar.

- Given a set X, by $\mathcal{P}(X)$ we denote the powerset of X.
- Given a set X, by $\mathcal{P}_{<\omega}(X)$ we denote the set of all finite subsets of X.
- Given sets X, Y, we write $X \Subset Y$ if X is a finite subset of Y.

2 Tukey-Teichmüller Theorem

In this section we observe the equivalence between Zorn's Lemma and the Tukey-Teichmüller theorem. We will compare the two statements and give a proof for the equivalence. Then we will see one of its natural applications, namely the proof that every vector space has a basis.

In the Introduction section we have already seen a statement of Zorn's Lemma and TT. These are very similar to each other: both statements require a certain condition on the order (poset vs. set system) of their concern, then conclude that a maximal element in that order exists. In the following lemma we will prove that the hypotheses of Zorn's Lemma and TT in some sense coincide. From this it will easily follow that ZL and TT are equivalent.

Lemma 2.1.

(a) For a non-empty system $\mathcal{A}$ of finite character on any set, the poset $(\mathcal{A}, \subseteq)$ has an upper bound for all chains.
(b) For any poset $(P, \leq)$ with an upper bound for all chains, the system $(\mathcal{P}, \subseteq)$ on P that consists of all chains of P is non-empty and has finite character.

Proof. First we prove (a). Since $\mathcal{A}$ is non-empty, the empty chain has an upper bound. Now, let $\mathcal{C}$ be a non-empty chain of $(\mathcal{A}, \subseteq)$. To prove $\bigcup \mathcal{C} \in \mathcal{A}$ using the finite character of $\mathcal{A}$, we show that every finite subset of $\bigcup \mathcal{C}$ belongs to $\mathcal{A}$. Let F be a finite subset of $\bigcup \mathcal{C}$. Then there exists $E \in \mathcal{C}$ with $F \Subset E \in \mathcal{A}$, so that $F \in \mathcal{A}$ by the finite character of $\mathcal{A}$. Therefore $\bigcup \mathcal{C} \in \mathcal{A}$. Now, each $C \in \mathcal{C}$ satisfies $C \subseteq \bigcup \mathcal{C}$, so that $\bigcup \mathcal{C}$ is an upper bound of $\mathcal{C}$ in $\mathcal{A}$. We conclude that every chain of $\mathcal{A}$ has an upper bound.

For (b), we begin by noting that since $\emptyset$ is a chain of P, $\mathcal{P}$ is non-empty. Now we prove that the system $\mathcal{P}$ has finite character, i.e. for all $S \subseteq P$, $S \in \mathcal{P}$ if and only if all finite subsets of S are in $\mathcal{P}$. 'only if ': Suppose $S \in \mathcal{P}$. If S' is a finite subset of S then, since S is a chain, S' is also a chain, hence $S' \in \mathcal{P}$. For 'if', we prove the contrapositive. Suppose that $S \notin \mathcal{P}$. Then S is no chain of P, so there are $p, q \in S$ s.t. $p \nleq q$ and $q \nleq p$. Then $\{p, q\} \notin \mathcal{P}$ because it is no chain of P. So not all finite subsets of S are in $\mathcal{P}$, as desired. This proves that $\mathcal{P}$ has finite character. □

Then we establish the equivalence.

Theorem 2.2. *Zorn's Lemma and the Tukey-Teichmüller Theorem are equivalent.*

Proof. First, we prove the Tukey-Teichmüller Theorem using Zorn's Lemma. Let $\mathcal{A}$ be a non-empty system on a set X. Then by Lemma 2.1(a), the poset $(\mathcal{A}, \subseteq)$ has an upper bound for each chain. Therefore, by Zorn's Lemma, we obtain a maximal element of $\mathcal{A}$ with respect to the inclusion, as desired.

Now, we prove the Zorn's Lemma using TT. Let $(P, \subseteq)$ be a poset with an upper bound for each chain. By Lemma 2.1(b), the system $(\mathcal{P}, \subseteq)$ on P that consists of all chains of P is non-empty and has finite character. Thus by TT, we obtain a maximal element M of $\mathcal{P}$. Since M is a chain of P, there is an upper bound m of M.

We claim that m is a maximal element of $(P, \subseteq)$. Let $x \in P$ s.t. $m \leq x$. Then $M \cup \{x\}$ is a chain of P and $M \subseteq M \cup \{x\}$. But M is maximal, so $M \cup \{x\} = M$. Therefore $x \in M$. Since m is an upper bound of M, $x \leq m$. Thus $x = m$, which proves that m is also maximal in $(P, \subseteq)$, as desired.

We conclude that Zorn's Lemma and the Tukey-Teichmüller Theorem are equivalent. □

Now we introduce the *enriched* form of the Tukey-Teichmüller Theorem, whose formulation slightly differs from the original, *plain* form of TT. While the plain form just says that *some* maximal element exists, the enriched form ensures the existence of a maximal element greater than a chosen element.

TT (enriched form). *Let X be a set, $\mathcal{A}$ a system of finite character on X. Then for all $A \in \mathcal{A}$, there is a maximal element $M \in \mathcal{A}$ such that $A \subseteq M$.*

Note that in the enriched form, we have relaxed the requirement of non-emptyness for $\mathcal{A}$. In the plain form we had to avoid the case $\mathcal{A} = \emptyset$ because otherwise the statement just becomes false. But in the enriched form, the case $\mathcal{A} = \emptyset$ is also alright, as we can see.

In the following we prove that in ZF, the two forms of TT are actually equivalent.

Theorem 2.3. *The plain form and the enriched form of TT are equivalent.*

Proof. If the enriched form holds, the plain version is clearly true, because we may invoke the enriched form on any $A \in \mathcal{A}$.

Now we prove the enriched form using the plain form. Let X, $\mathcal{A}$ and A be as in the statement. Define

$$\mathcal{B} = \{B \subseteq X : B \cup A \in \mathcal{A}\}.$$

Note that $\mathcal{B}$ contains all supersets $A' \in \mathcal{A}$ of A because $A' \cup A = A' \in \mathcal{A}$.

We prove that the system $\mathcal{B}$ on X has finite character, i.e. for all $B \subseteq X$, $X \in \mathcal{B}$ if and only if all finite subsets of B are in $\mathcal{B}$. 'only if': Suppose $B \in \mathcal{B}$. Let F be a finite subset of B. Then $B \cup A \in \mathcal{A}$. Since $F \cup A \subseteq B \cup A$ and so all finite subsets of $F \cup A$ are finite subsets of $B \cup A$, by the finite character of $\mathcal{A}$, we conclude that $F \cup A \in \mathcal{A}$. So $F \in \mathcal{B}$, as desired. For 'if', we prove the contrapositive. Suppose $B \notin \mathcal{B}$. We look for a finite subset of B that is not in $\mathcal{B}$. Then $B \cup A \notin \mathcal{A}$. Thus there is a finite subset $B' \cup A'$ of $B \cup A$, where $B' \Subset B$ and $A' \Subset A$, such that $B' \cup A' \notin \mathcal{A}$. Since $B' \cup A' \Subset B' \cup A$, by the finite character of $\mathcal{A}$, we also have that $B' \cup A \notin \mathcal{A}$. Therefore $B' \notin \mathcal{A}$; this proves 'if' because $B' \Subset B$.

We note that $A \in \mathcal{B} \neq \emptyset$ because $A \cup A = A \in \mathcal{A}$. Thus we may apply the plain TT to $\mathcal{B}$ and obtain a maximal element $M \in \mathcal{B}$. It follows that $M \cup A \in \mathcal{A}$ and so $(M \cup A) \cup A \in \mathcal{A}$. Since M is maximal in $\mathcal{B}$ and $M \subseteq M \cup A$, we have that $M = M \cup A$ and so $A \subseteq M$. Now, since all supersets of A in $\mathcal{A}$ are in $\mathcal{B}$ and $M \supseteq A$ is maximal in $\mathcal{B}$, M is also maximal in $\mathcal{A}$, as desired.

We conclude that the plain form indeed implies the enriched form. □

We will often encounter this phenomenon of the equivalent plain and enriched forms. In fact, as well as the restrictions of TT we will discuss in the following sections, Zorn's Lemma can be enriched in a similar manner.

Since our main concerns are the principles in systems of finite character, we will not state the enriched form of Zorn's Lemma nor prove its equivalence here. (One may look at [3] for a proof.) However, the proof can be made using a similar strategy as we did for Theorem 2.3.

Now we turn to an application of TT. As promised in the beginning of this section, we will prove the following important result in Linear Algebra.

Proposition 2.4. *Assume TT. Then every vector space has a basis.*

Proof. Let V be any vector space, and $\mathcal{A}$ be the system on V consisting of all linearly independent subsets of V. We show that $\mathcal{A}$ has finite character to invoke TT and obtain a maximal linearly independent subset of V. Then that subset will turn out to be a basis of V.

We show that $\mathcal{A}$ has finite character, i.e. for all $A \subseteq V$, $A \in \mathcal{A}$ if and only if all finite subsets of A are in $\mathcal{A}$. 'only if': If $A \in \mathcal{A}$, then any finite subset of A is also linearly independent and hence in $\mathcal{A}$. For 'if', we prove the contrapositive. Suppose $A \notin \mathcal{A}$. Then there exist linearly dependent vectors $v_1, \ldots, v_n \in A$, with $n \in \mathbf{Z}_{\geq 2}$. So the finite subset $\{v_1, \ldots, v_n\}$ of A is not in $\mathcal{A}$, as desired. Thus $\mathcal{A}$ has finite character.

By the Tukey-Teichmüller Theorem, we obtain a maximal linearly independent subset B of V. To establish that B is a basis of V, we show that B generates V. Suppose B doesn't, i.e. there is $v \in V$ which cannot be written as a linear combination of B. It means that v is linearly independent of B, and contradicts B being maximal in $\mathcal{A}$. Therefore B must generate V and so B is a basis of V.

We conclude that every vector space has a basis. □

3 Boolean Algebra and PIT

In this section we review Boolean Algebra and discuss the Boolean Prime Ideal Theorem and the Ultrafilter Lemma.

3.1 Boolean Algebra and PIT

Before we discuss about the Boolean Prime Ideal Theorem, we review some basic aspects about Boolean Algebra. First we define Boolean Algebra.

Definition 3.1. A *Boolean Algebra* is a structure $(B, \vee, \wedge, ', 0, 1)$, where B is a set, $\vee, \wedge$ are binary operations on B, $'$ is an unary operation on B and $0, 1 \in B$, satisfying

- $a \vee a = a$ and $a \wedge a = a$, (idempotency)
- $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, (associativity)
- $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$, (commutativity)
- $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$, (absorption)

- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, (distributivity)
- $0 \vee a = a$ and $0 \wedge a = a$, (bottom boundary)
- $1 \vee a = 1$ and $1 \wedge a = a$, (top boundary)
- $a \vee a' = 1$ and $a \wedge a' = 0$, (complements)

for all $a, b, c \in B$.

Example 3.2. A typical example of Boolean algebra is the Powerset algebra. For any set X, we the structure $(\mathcal{P}(X), \cup, \cap, (-)^c, \emptyset, X)$ is a Boolean algebra that we refer to as the *Powerset algebra* on X. Notice that the unary operator $(-)^c$ is defined as $Y^c = X - Y$ for all $Y \in \mathcal{P}(X)$. Also in this paper, we will encounter Powerset algebra; at the end of this section when we derive the Axiom of Choice for Finite Sets from the Ultrafilter Lemma, and in Section 4 when we prove that UL implies the Restricted Tukey-Teichmüller Theorem.

To be able to formulate the statement of Boolean Prime Ideal Theorem, we need the definition of Boolean ideal and Boolean prime ideal. As the name suggests, ideals of Boolean Algebra are similar to ideals of rings. In the following we will also define the *finite join property* for subsets of Boolean Algebras; this concept will play an important role when we later prove the Prime Ideal Theorem using other non-constructive principles.

Definition 3.3. An ideal of a Boolean Algebra B is a subset J of B such that

(1) $0 \in J$ and $1 \notin J$,
(2) if $a, b \in J$ then $a \vee b \in J$,
(3) if $a \in J$ and $b \in B$ then $a \wedge b \in J$.

And an ideal J that satisfies

- if $a, b \in B$ with $a \wedge b \in J$ then $a \in J$ or $b \in J$

is called a *prime ideal.*

We say that a subset J has the *finite join property* if $a_1 \vee \cdots \vee a_n \neq 1$ for all $a_1, \ldots, a_n \in J$.

In the following lemma we state several conditions about ideals and prime ideals that are useful.

Lemma 3.4. *Let J be a subset of a Boolean Algebra B. Then the following hold.*

(a) Suppose that J is an ideal. Then J has the finite join property.
(b) Suppose that J is a prime ideal. Then $a \in J$ or $a' \in J$ for all $a \in J$.

(c) *J satisfies the finite join property and either $a \in J$ or $a' \in J$ for all $a \in B$ if and only if J is a prime ideal.*

Proof. First we prove (a). Let $b_1, \dots, b_n$ be finitely many elements of I. Since I is an ideal, $b_1 \vee \dots \vee b_n \in I$, so that $b_1 \vee \dots \vee b_n \neq 1$ as desired.

Now we prove (b). Let $a \in B$. Then $a \wedge a' = 0 \in I$, because I is an ideal. Since I is prime, $a \in I$ or $a' \in I$, as desired.

Lastly we prove (c). 'if': By (a) and (b), J has finite join property and $a \in J$ or $a' \in J$ for all $a \in J$, as desired. 'only if': Suppose, for contradiction. That there are $a, b \in B$ such that $a \wedge b \in J$ but $a \notin J$ and $b \notin J$. Then, by the hypothesis, $a' \in J$. Thus $(a \wedge b) \vee a' \vee b' = ((a \vee a') \wedge (b \vee a')) \vee b' = (1 \wedge (b \vee a')) \vee b' = (1 \vee b') \wedge (b \vee a' \vee b') = 1 \wedge (1 \vee a') = 1 \wedge 1 = 1$, contradicting the finite join property. Therefore J must be a prime ideal. $\square$

Now we state the Boolean Prime Ideal Theorem. Thereafter we will see that PIT is indeed a theorem in ZFC; the Tukey-Teichmüller theorem implies it.

Boolean Prime Ideal Theorem (PIT). *Every ideal of a Boolean algebra is included in a prime ideal.*

To prove PIT using a non-constructive principle for systems of finite character, e.g. the TT or the RTT to come later, it is useful to formulate the following lemma.

Lemma 3.5 (extension property for finite join property)**.** *Let B be a Boolean algebra and $J \subseteq B$. If J has the finite join property and $x \in B$, then $J \cup \{x\}$ or $J \cup \{x'\}$ has the finite join property.*

Proof. Suppose, for contradiction, that neither $J \cup \{x\}$ nor $J \cup \{x'\}$ has the finite join property. Since J has finite join property, there exist $a_1, \dots, a_n \in J$ and $b_1, \dots, b_k \in J$ such that

$$a_1 \vee \dots \vee a_n \vee x = 1 \text{ and } b_1 \vee \dots \vee b_m \vee x' = 1.$$

If we put

$$z = a_1 \vee \dots \vee a_n \vee b_1 \vee \dots \vee b_m,$$

then $z \vee x = 1$ and $z \vee x' = 1$ hold. Therefore $1 = 1 \wedge 1 = (z \vee x) \wedge (z \vee x') = (z \wedge z) \vee (z \wedge x') \vee (x \wedge z) \vee (x \wedge x') = z \vee 0 = z$; this is impossible because $z \neq 1$ by the finite join property of J.

We conclude that either $J \cup \{x\}$ or $J \cup \{x'\}$ must have the finite join property. $\square$

Remark 3.6. We anticipate that (a slight generalization of) this extension property will be the (only) extra hypothesis to add to the statement of the Tukey-Teichmüller Theorem when we formulate the Restricted TT. We will also weaken the conlusion of TT slightly, but such that it is still strong enough to imply the conclusion of PIT. Therefore, once we formulated the RTT, we will be able to replace the use of TT in the following proof by RTT.

Theorem 3.7. *The Tukey-Teichmüller Theorem implies PIT.*

Proof. Let I be an ideal of a Boolean Algebra B. Define

$$\mathcal{J} = \{J \subseteq B : J \text{ has the finite join property}\}.$$

We prove that the system $\mathcal{J}$ on B has finite character, i.e. for all $J \subseteq B$, $J \in \mathcal{J}$ if and only if all finite subsets of J are in $\mathcal{J}$. 'only if': Suppose $J \in \mathcal{J}$. Then J and hence all subsets of J have the finite join property. So all finite subsets of J are in $\mathcal{J}$. For 'if', we prove the contrapositive. Suppose $J \notin \mathcal{J}$. Then J does not have the finite join property. This means that there exist $b_1, \dots, b_n \in J$ such that $b_1 \vee \dots \vee b_n = 1$. Thus $\{b_1, \dots, b_n\} \in \mathcal{J}$, as desired. Therefore $\mathcal{J}$ has finite character.

Note that by Lemma 3.4(a), I has finite join property and hence $I \in \mathcal{J}$. Thus by TT, there exists $M \in \mathcal{J}$ such that $I \subseteq M$ and M is maximal in $\mathcal{J}$. In particular, M has finite join property.

Let $x \in B$. By Lemma 3.5, $M \cup \{x\}$ or $M \cup \{x'\}$ has the finite join property and hence in $\mathcal{J}$. So by the maximality of M in $\mathcal{J}$, $x \in M$ or $x' \in M$. By Lemma 3.4(c) we find that M is a desired prime ideal.

We conclude that PIT follows from the Tukey-Teichmüller Theorem. □

Remark 3.8. Theorem 3.7 also says, since TT and the Axiom of Choice are equivalent, that AC implies PIT. This implications is strict, i.e. PIT does *not* imply AC in ZF; cf. [4].

3.2 Ultrafilter Lemma and its applications

We can dualize everything we have written about ideals. Then we obtain the notion of *filters* and *ultrafilters*, the dual concepts of ideals and prime ideals, as well as the *Ultrafilter Lemma*, which is the dual statement of the Prime Ideal Theorem.

Definition 3.9. A filter of a Boolean Algebra B is a subset J of B such that

(1) $0 \notin J$ and $1 \in J$,
(2) if $a, b \in J$ then $a \wedge b \in J$,
(3) if $a \in J$ and $b \in B$ then $a \vee b \in J$.

And a filter J that satisfies

- if $a \vee b \in J$ then $a \in J$ or $b \in J$

is called an *ultrafilter*.

We say that a subset J has the *finite meet property* if $a_1 \wedge \cdots \wedge a_n \neq 0$ for all $a_1, \ldots, a_n \in J$.

In the following we will present the properties about filters and ultrafilters, analogous to those of ideals and prime ideals that we have discussed. Since the properties have already been proved for the ideal case, for the filter case we shall not need to prove them again from the scratch, thanks to the well-known *duality principle*:

Proposition 3.10 (Duality Principle)**.** *Let Φ be a statement about Boolean algebras, involving the symbols $\vee, \wedge, ', 0, 1$. Define the dual statement Φ^d of Φ by interchanging $\vee$ with $\wedge$ and 0 with 1 in Φ. If Φ holds for all Boolean algebras, then Φ^d is also true for all Boolean algebras.*

Proof. See, for instance, [3]. □

Lemma 3.11. *Let J be a subset of a Boolean Algebra B. Then the following hold.*

(a) Suppose that J is a filter. Then J has the finite meet property.
(b) Suppose that J is an ultrafilter. Then $a \in J$ or $a' \in J$ for all $a \in J$.
(c) J satisfies the finite meet property and either $a \in J$ or $a' \in J$ for all $a \in B$ if and only if J is an ultrafilter.

Proof. These statements is dual to those of Lemma 3.4. Therefore, by the duality principle, (a), (b) and (c) are true. □

Ultrafilter Lemma (UL). *Every filter of a Boolean algebra is included in an ultrafilter.*

Lemma 3.12. *PIT and the Ultrafilter Lemma are equivalent.*

Proof. This follows from the duality principle and the fact that PIT is dual to UL. □

In other words, Ultrafilter Lemma holds in settings where PIT holds, and vice versa. In section 4, we will make use of this fact; we will show that the Ultrafilter Lemma proves the Restricted Tukey-Teichmüller Theorem (RTT) to establish PIT$\Rightarrow$RTT.

Before we discuss the Axiom of Choice for Finite Sets, we first prove two lemmas regarding special properties of filters and ultrafilters in a Powerset algebra. These properties will be convenient when we need to apply the Ultrafilter Lemma for a Powerset algebra.

The first lemma says that every collection with the *finite intersection property*, i.e. the finite meet property for Powerset algebras, is included in a filter. This allows one to invoke the Ultrafilter Lemma on a non-empty collection with the finite intersection property, without requiring the collection to be a filter.

Lemma 3.13. *Let X be a set and $\mathcal{A} \subseteq \mathcal{P}(X)$ a non-empty collection with the finite intersection property. Then there exists a filter $\hat{\mathcal{A}}$ of $\mathcal{P}(X)$ such that $\mathcal{A} \subseteq \hat{\mathcal{A}}$.*

Proof. Define

$$\hat{\mathcal{A}} = \{A \cup K : A \in \mathcal{A}, K \in \mathcal{P}(X)\}.$$

Then clearly $\mathcal{A} \subseteq \hat{\mathcal{A}}$ because $A = A \cup \emptyset \in \hat{\mathcal{A}}$ for all $A \in \mathcal{A}$. It remains to verify that $\hat{\mathcal{A}}$ satisfies (1), (2) and (3) of the filter definition in 3.9.

First we verify (1). Note that $\emptyset \notin \mathcal{A}$ because $\mathcal{A}$ has finite intersection property. Therefore each element $A \cup K \in \hat{\mathcal{A}}$ (where $A \in \mathcal{A}$ and $K \in \mathcal{P}(X)$) is non-empty. Now, since $\mathcal{A}$ is non-empty, choose $A \in \mathcal{A}$ arbitrary. Then $X = A \cup X \in \hat{\mathcal{A}}$. We conclude that $\emptyset \notin \hat{\mathcal{A}}$ and $X \in \hat{\mathcal{A}}$, as desired.

For (2), let $A \cup K, B \cup L \in \hat{\mathcal{A}}$ where $A, B \in \mathcal{A}$ and $K, L \in \mathcal{P}(X)$. Then as desired,

$$(A \cup K) \cap (B \cup L) = (A \cap B) \cup (A \cap Y) \cup (X \cap B) \cup (X \cap Y) \in \hat{\mathcal{A}}$$

because $(A \cap B) \in \mathcal{A}$ and $(A \cap Y) \cup (X \cap B) \cup (X \cap Y) \in \mathcal{P}(X)$.

For (3), let $A \cup K \in \hat{\mathcal{A}}$ and $L \in \mathcal{P}(X)$. Then as desired,

$$(A \cup K) \cup L = A \cup (K \cup L) \in \hat{\mathcal{A}},$$

because $A \in \mathcal{A}$ and $(K \cup L) \in \mathcal{P}(X)$.

This proves that $\hat{\mathcal{A}}$ is a filter of $\mathcal{P}(X)$ with $\mathcal{A} \subseteq \hat{\mathcal{A}}$. □

The second lemma describes a very useful property of an ultrafilter in a Powerset algebra.

Lemma 3.14. *Let $\mathcal{U}$ be an ultrafilter of the Powerset algebra on a set X. If $X_1, \ldots, X_n \subseteq X$ are pairwise disjoint and $X = X_1 \cup \cdots \cup X_n$, then there is a unique $1 \leq k \leq n$ such that $X_k \in \mathcal{U}$.*

Proof. The existence part is proved as follows. Suppose that for all $1 \leq k \leq n$, $X_k \notin \mathcal{U}$. Then $X_1^c, \ldots, X_n^c \in \mathcal{U}$. But $X_1^c \cap \cdots \cap X_n^c = (X_2 \cup \cdots \cup X_n) \cap \cdots \cap (X_1 \cup \cdots \cup X_{n-1}) = \emptyset$, contradicting Lemma 3.11(a) that $\mathcal{U}$ has the finite intersection(meet) property. This proves the existence part.

The uniqueness part is easy; if k, k' ($1 \leq k, k' \leq n$) are such that $k \neq k'$ but $X_k, X_{k'} \in \mathcal{U}$, then $\emptyset = X_k \cap X_{k'} \in \mathcal{U}$, which is impossible because $\mathcal{U}$ has the finite intersection property and so $\emptyset \notin \mathcal{U}$. □

Now, using the two results just obtained, we will prove that the Ultrafilter Lemma implies the Axiom of Choice for Finite Sets. This result of the

Ultrafilter Lemma will again be referred to when we derive the Restricted Tukey-Teichmüller Theorem from PIT in Section 4.

Axiom of Choice for Finite Sets. *Let* $\mathcal{Z} = \{Z_t : t \in T\}$ *be a collection of non-empty* finite *sets indexed by a set* T*. Then there exists a choice function for* $\mathcal{Z}$*, i.e. a function* $\phi : T \to \bigcup \mathcal{Z}$ *s.t.* $\phi(t) \in Z_t$ *for all* $t \in T$*.*

Theorem 3.15. *The Ultrafilter Lemma implies the Axiom of Choice for Finite Sets.*

Proof. Let $\mathcal{Z} = \{Z_t : t \in T\}$ be a collection of non-empty finite sets. A *partial choice function* for $\mathcal{Z}$ is a function $\phi : S \to \bigcup \mathcal{Z}$ with $S \subseteq T$ and $\phi(s) \in Z_s$ for all $s \in S$. Note that for all *finite* subset F of T, there is a partial choice function whose domain is F; we can construct such finite choice functions by induction on the cardinality of F. Define E as the set of all partial choice functions for $\mathcal{Z}$. And for each $F \Subset T$, define

$$E_F = \{\phi \in E : F \subseteq \mathrm{dom}(\phi)\}.$$

Note (a) that $E_F \neq \emptyset$ for all $F \Subset T$ because there is a $\phi \in E_F$ with $\mathrm{dom}(\phi) = F$.

We also claim (b) that for $F, G \Subset T$, $E_{F\cup G} \subseteq E_F \cap E_G$. Let $\phi \in E_{F\cup G}$. Then $F \cup G \subseteq \mathrm{dom}(\phi)$. So $F \subseteq \mathrm{dom}(\phi)$ and $G \subseteq \mathrm{dom}(\phi)$ hold. Therefore $\phi \in E_F \cap E_G$, as desired.

From (a) and (b) it follows that the collection non-empty $\mathcal{A} := \{E_F : F \Subset T\}$ has the finite intersection property, because if $F_1, \ldots, F_n \Subset T$ then $E_{F_1} \cap \cdots \cap E_{F_n} \supseteq E_{F_1\cup\cdots\cup F_n} \neq \emptyset$. So, by Lemma 3.13 and the Ultrafilter Lemma, we obtain an ultrafilter $\mathcal{U}$ of $\mathcal{P}(E)$ such that $\mathcal{A} \subseteq \mathcal{U}$.

Let $t \in T$ and write $Z_t = \{z_1, \ldots, z_n\}$. Then $E_{\{t\}} = E_{\{t\},1} \cup \cdots \cup E_{\{t\},n}$, where for $1 \leq k \leq n$,

$$E_{\{t\},k} := \{\phi \in E_{\{t\}} : \phi(t) = z_k\}.$$

We have that $E_{\{t\},k} \cap E_{\{t\},k'} = \emptyset$ for $1 \leq k < k' \leq n$, because if $\phi \in E_{\{t\},k}$ then $\phi(t) = z_k \neq z_{k'}$ so $\phi \notin E_{\{t\},k'}$. By Lemma 3.14, there is a unique integer $k(t)$, $1 \leq k(t) \leq n$, such that $E_{\{t\},k(t)} \in \mathcal{U}$. Finally, define the choice function $\Phi : T \to \bigcup \mathcal{Z}$ by $\Phi(t) = z_{k(t)}$, as desired. □

4 Restricted Tukey-Teichmüller Theorem

In this section we discuss the restriction of the Tukey-Teichmüller Theorem, RTT, given by Hodel [1]. As anticipated before, the restriction will turn out to be equivalent to the Boolean Prime Ideal theorem (PIT).

We will first state RTT and see what it can do, including a proof that RTT implies Alexander's Subbase Theorem, an important result in Topology. Then the two variations RTT^+ and RTT^{++} of RTT will be formulated, which we will primarily use to bridge the equivalence of RTT and PIT. Finally, several selection lemmas due to Cowen, Engeler and Rado will be treated as applications of RTT^{++}.

4.1 Statement of RTT

Restricted Tukey-Teichmüller Theorem (RTT). *Let X be a set, $\mathcal{A}$ a (non-empty) system of finite character on X, and let $'$ be an unary operation on X. Assume that*

(E) *$\mathcal{A}$ has the extension property with respect to $'$, i.e. for all $A \in \mathcal{A}$ and all $x \in X$, $A \cup \{x\} \in \mathcal{A}$ or $A \cup \{x'\} \in \mathcal{A}$.*

Then

(I) there exists $B \in \mathcal{A}$ such that for all $x \in X$, $x \in B$ or $x' \in B$,
(II) for all $A \in \mathcal{A}$, there exists $B \in \mathcal{A}$ such that $A \subseteq B$ and for all $x \in X$, $x \in B$ or $x' \in B$.

(I) and (II) are, respectively, the conclusions of the *plain form* and the *enriched form* of RTT. For the enriched form of RTT, we do not require the system $\mathcal{A}$ to be non-empty.

Like in the case of the original Tukey-Teichmüller Theorem, the two forms are equivalent in ZF. Note that the strategy used for the proof of the following theorem is the same as that of Theorem 2.3.

Theorem 4.1. *The plain form and the enriched form of RTT are equivalent.*

Proof. If the enriched form holds, the plain version is clearly true, because we may invoke the enriched form on any $A \in \mathcal{A}$.

Now we prove the enriched form using the plain form. Let X, $\mathcal{A}$ and A be as in the statement. Note that this time, deviating from the case of Theorem 2.3, $\mathcal{A}$ additionally satisfies (E). Define

$$\mathcal{B} = \{B \subseteq X : B \cup A \in \mathcal{A}\}.$$

Note that $\mathcal{B}$ contains all supersets $A' \in \mathcal{A}$ of A because $A' \cup A = A' \in \mathcal{A}$.

We prove that the system $\mathcal{B}$ on X has finite character, i.e. for all $B \subseteq X$, $X \in \mathcal{B}$ if and only if all finite subsets of B are in $\mathcal{B}$. (This part is exactly the same as in the proof of Theorem 2.3.) 'only if': Suppose $B \in \mathcal{B}$. Let F be a finite subset of B. Then $B \cup A \in \mathcal{A}$. Since $F \cup A \subseteq B \cup A$ and so all finite subsets of $F \cup A$ are finite subsets of $B \cup A$, by the finite character of $\mathcal{A}$, we conclude that $F \cup A \in \mathcal{A}$. So $F \in \mathcal{B}$, as desired. For 'if', we prove the contrapositive. Suppose $B \notin \mathcal{B}$. We look for a finite subset of B that is not in $\mathcal{B}$. Then $B \cup A \notin \mathcal{A}$. Thus there is a finite subset $B' \cup A'$ of $B \cup A$, where $B' \Subset B$ and $A' \Subset A$, such that $B' \cup A' \notin \mathcal{A}$. Since $B' \cup A' \Subset B' \cup A$, by the finite character of $\mathcal{A}$, we also have that $B' \cup A \notin \mathcal{A}$. Therefore $B' \notin \mathcal{A}$; this proves 'if' because $B' \Subset B$.

Then we check that $\mathcal{B}$ has the extension property, i.e. for all $B \in \mathcal{B}$ and all $x \in X$, $B \cup \{x\} \in \mathcal{B}$ or $B \cup \{x'\} \in \mathcal{B}$. Let $x \in X$ and $B \in \mathcal{B}$. Since $\mathcal{A}$ has the extension property, we know that $(B \cup A) \cup \{x\} \in \mathcal{A}$ or $(B \cup A) \cup \{x'\} \in \mathcal{A}$. In other words, $(B \cup \{x\}) \cup A \in \mathcal{A}$ or $(B \cup \{x'\}) \cup A \in \mathcal{A}$. This proves that $B \cup \{x\} \in \mathcal{B}$ or $B \cup \{x'\} \in \mathcal{B}$, as desired.

We note that $A \in \mathcal{B} \neq \emptyset$ because $A \cup A = A \in \mathcal{A}$. Thus we may apply the plain RTT to $\mathcal{B}$ and obtain an element $M \in \mathcal{B}$ such that $x \in M$ or $x' \in M$ for all $x \in X$. Since $A \cup M \in \mathcal{A}$, $A \subseteq A \cup M$ and $x \in A \cup M$ or $x' \in A \cup M$ for all $x \in X$, we have that $A \cup M$ is a desired element we sought.

We conclude that the plain form indeed implies the enriched form. $\square$

Recall the proof of Theorem 3.7. When we derived PIT using TT, we deduced from the maximality of the element M obtained by TT, the property from RTT that $x \in M$ or $x' \in M$ for all $x \in B$. In other words, for deriving PIT, the maximality property of M is unnecessarily strong while the property from RTT is strong enough, as anticipated in Remark 3.6. Since the finite join property has the extension property (Lemma 3.5), we could indeed apply RTT instead of TT. This proves that also RTT implies PIT; in the following we will work this out.

Theorem 4.2. *The Restricted Tukey-Teichmüller Theorem implies the Boolean Prime Ideal Theorem.*

Proof. Let I be an ideal of a Boolean Algebra B. Define

$$\mathcal{J} = \{J \subseteq B : J \text{ has the finite join property}\}.$$

In the proof of Theorem 3.7, we have seen that $\mathcal{J}$ has finite character and $I \in \mathcal{J}$. And by Lemma 3.5, $\mathcal{J}$ has the extension property w.r.t. the operation $'$ on B. So we may apply RTT on $\mathcal{J}$, and obtain $M \in \mathcal{J}$ such that $I \subseteq M$ and $x \in M$ or $x' \in M$ for all $x \in B$. By Lemma 3.4(c) we find that M is a desired prime ideal.

We conclude that PIT follows from the Restricted Tukey-Teichmüller Theorem. $\square$

Then we turn to the study of Alexander's Subbase Theorem.

Alexander's Subbase Theorem. *Let X be a topological space and $\mathcal{S}$ be a subbase for X such that every cover $\mathcal{C} \subseteq \mathcal{S}$ has a finite subcover. Then X is compact.*

To exhibit the relation of this principle with RTT, we introduce the terminology of *finite inadequacy.*

Definition 4.3. If X is a topological space, we say that a collection $\mathcal{W} \subseteq \mathcal{P}(X)$ is *finitely inadequate* if no finite subcollection of $\mathcal{W}$ covers X.

The key of the proof that RTT implies Alexander's Subbase Theorem is that the property of finite inadequacy satisfies the conditions (I) and (II) in the hypothesis of RTT. The following lemma explains this in detail.

Lemma 4.4. *Let $\mathscr{A}$ be the system on $\mathcal{P}(X)$ consisting of all finitely inadequate systems on X. Then*

(a) $\mathscr{A}$ has finite character,
(b) $\mathscr{A}$ has the extension property w.r.t. the complement operation $(-)^c := X - (-)$.

Proof. First we prove (a) that for all $\mathcal{A} \subseteq \mathcal{P}(X)$, $\mathcal{A} \in \mathscr{A}$ if and only if all finite subcollections of $\mathcal{A}$ are in $\mathscr{A}$. 'if': Suppose that $\mathcal{A} \in \mathscr{A}$. In other words, no finite subcollection of $\mathcal{A}$ covers X. Therefore, if $\mathcal{F} \Subset \mathcal{A}$, then no finite subcollection of $\mathcal{F}$ covers X and hence $\mathcal{F} \in \mathscr{A}$, as desired. For 'only if', we prove the contrapositive. Suppose that $\mathcal{A} \notin \mathscr{A}$. Then there is a finite subcollection $\mathcal{F}$ of $\mathcal{A}$ that covers X. Since $\mathcal{F}$ is a finite subcollection of itself that covers X, $\mathcal{F}$ is *not* finitely inadequate and hence in $\mathscr{A}$. Thus not every finite subcollection of $\mathcal{A}$ is in $\mathscr{A}$, as desired.

For (b), let $\mathcal{A} \in \mathscr{A}$ and $Y \in \mathcal{P}(X)$. We prove that $\mathcal{A} \cup \{Y\} \in \mathscr{A}$ or $\mathcal{A} \cup \{Y^c\} \in \mathscr{A}$. Suppose, for contradiction, that $\mathcal{A} \cup \{Y\} \notin \mathscr{A}$ and $\mathcal{A} \cup \{Y^c\} \notin \mathscr{A}$. This means, since $\mathcal{A}$ does not have a subcollection that covers X, that there are $\mathcal{F}, \mathcal{G} \Subset \mathcal{A}$ such that $\mathcal{F} \cup \{Y\}$ covers X and $\mathcal{F} \cup \{Y^c\}$ covers X. Thus $(\mathcal{F} \cup \mathcal{G}) \cup \{Y\}$ covers X and $(\mathcal{F} \cup \mathcal{G}) \cup \{Y^c\}$ covers X. In other words, $\bigcup(\mathcal{F} \cup \mathcal{G}) \cup Y = X$ and $\bigcup(\mathcal{F} \cup \mathcal{G}) \cup Y^c = X$. Since $\mathcal{F} \cup \mathcal{G}$ is finite and so cannot cover X, we have that $(\bigcup(\mathcal{F} \cup \mathcal{G}))^c \neq \emptyset$, $(\bigcup(\mathcal{F} \cup \mathcal{G}))^c \subseteq Y$ and $(\bigcup(\mathcal{F} \cup \mathcal{G}))^c \subseteq Y^c$, which gives a contradiction as desired. □

Remark 4.5. Note that the part (b) of the proof, which proves the extension property for finite inadequacy, is very similar to the proof of Lemma 3.5, which proves the extension property for finite join property. The explanation for this is that the property of finite inadequacy, which requires $A_1 \cup \cdots \cup A_n \neq X$ for $A_1, \ldots, A_n \in Y$, and the operation ^{c} are respectively analogous to the finite meet property, which requires $a_1 \vee \cdots \vee a_n \neq 1$ for $a_1, \ldots, a_n \in B$, and the operation $'$. This analogy becomes clearer if we recall that $\mathcal{P}(X)$ can be regarded as a Boolean algebra with $\cup$ as the meet operation and ^{c} as the complement operation.

Now we give the proof that the Alexander's Subbase Theorem follows from the Restricted Tukey-Teichmüller Theorem.

Lemma 4.6. *RTT implies Alexander's Subbase Theorem.*

Proof. Let X be a topological space and $\mathcal{S}$ be a subbase for X such that every cover included in $\mathcal{S}$ has a finite subcover. Suppose, for contradiction, that X is not compact. Then there is an open cover $\mathcal{W}$ of X such that there is no finite subcollection of $\mathcal{W}$ covers X. In other words, $\mathcal{W}$ is finitely inadequate. Define

$$\mathcal{B} = \{B : B \subseteq W \text{ for } W \in \mathcal{W} \text{ and } B = S_1 \cap \cdots \cap S_n \text{ for } S_1, \ldots, S_n \in \mathcal{S}\}.$$

To prove that $\mathcal{B}$ covers X, we show that for each $W \in \mathcal{W}$ there exists a subcollection $\mathcal{B}_W$ of $\mathcal{B}$ such that $W \subseteq \bigcup \mathcal{B}_W$. For each $w \in W$, there exist $S_1, \ldots, S_n \in \mathcal{S}$ such that $w \in S_1 \cap \cdots \cap S_n \subseteq W$, because W is open and $S_1 \cap \cdots \cap S_n$ is a member of the basis generated by the subbase $\mathcal{S}$. Thus, put $\mathcal{B}_W = \{B = S_1 \cap \cdots \cap S_n : n \in \mathbf{N} \wedge \exists w \in W(w \in B) \wedge B \subseteq W\}$; this is a subcollection of $\mathcal{B}$ with $W \subseteq \bigcup \mathcal{B}_W$, as desired. Now, since $\mathcal{W}$ covers X, also $\mathcal{B} \supseteq \bigcup_{W \in \mathcal{W}} \mathcal{B}_W$, which covers each $W \in \mathcal{W}$, covers X.

Now we prove that $\mathcal{B}$ is finitely inadequate. Suppose it isn't, i.e. there is a finite subcollection $\mathcal{B}'$ of $\mathcal{B}$ that covers X. Since $\mathcal{B}'$ is finite, in plain ZF, we can choose for each $B \in \mathcal{B}'$ a $W_B \in \mathcal{W}$ such that $B \subseteq W_B$. Then the finite subcollection $\{W_B : B \in \mathcal{B}'\}$ of $\mathcal{W}$ covers X, contradicting $\mathcal{W}$ being finitely inadequate. Therefore $\mathcal{B}$ must be finitely inadequate.

Let $\mathscr{A}$ be the system consisting of all finitely inadequate systems on X. By Lemma 4.4, $\mathscr{A}$ satisfies the hypothesis of RTT. Since $\mathcal{B} \in \mathcal{A}$, by the enriched RTT, there exists $\mathcal{M} \in \mathscr{A}$ such that

(1) $\mathcal{B} \subseteq \mathcal{M}$,
(2) for all $Y \subseteq X$, $Y \in \mathcal{M}$ or $Y^c \in \mathcal{M}$.

We prove (3) that for each $B \in \mathcal{B}$, there exists $S \in \mathcal{M} \cap \mathcal{S}$ such that $B \subseteq S$. Let $B \in \mathcal{B}$. Then $B = S_1 \cap \cdots \cap S_n$ for $S_1, \ldots, S_n \in \mathcal{S}$. We show that there is k, $1 \leq k \leq n$, such that $S_k \in \mathcal{M}$. Suppose there isn't, i.e. $S_1, \ldots, S_n \notin \mathcal{M}$. From (2) it follows that $S_1^c, \ldots, S_n^c \in \mathcal{M}$. Then $X = B \cup B^c = B \cup (S_1 \cap \cdots \cap S_n)^c = B \cup (S_1^c \cup \cdots \cup S_n^c)$; in other words, the finite subcollection $\{B, S_1^c, \ldots, S_n^c\}$ of $\mathcal{M}$ cover X, contradicting $\mathcal{M}$ being finitely inadequate. So there exists k, $1 \leq k \leq n$, such that $S_k \in \mathcal{M}$. Since $S_k \in \mathcal{S}$ as well, we have that $S_k \in \mathcal{M} \cap \mathcal{S}$ as desired.

By (3), $\mathcal{M} \cap \mathcal{S} \subseteq \mathcal{S}$ covers X because $\mathcal{B}$ covers X. Then by the hypothesis, there is a finite subcollection $\mathcal{F}$ of $\mathcal{M} \cap \mathcal{S}$ that covers X. This is a contradiction because $\mathcal{M} \supseteq \mathcal{F}$ is finitely inadequate. Therefore X must be compact.

This proves that RTT implies Alexander's Subbase Theorem. □

Remark 4.7. This proof has slightly been optimized from the original proof in [1]. In particular, we have not invoked the Axiom of Choice for Finite Sets, while the original proof used it.

4.2 Equivalence to PIT

In the previous section, we have seen that RTT implies PIT. In this section we will finish the equivalence between the two. To bridge the inverse implication, Hodel [1] introduces the two variations RTT^+ and RTT^++ of RTT. RTT^+ will be considered as a generalization of RTT, and RTT^{++} of RTT^+. (But they will turn out to be still equivalent to RTT.) However, apart from their role to connect PIT and RTT, each of them will have its own characteristic applications; we will see these applications in the subsequent section.

RTT^+. *Let X be a set, $\mathcal{A}$ a non-empty system of finite character on X and $\mathcal{Z} = \{Z_t : t \in T\}$ a collection of finite subsets of X indexed by some set T. Assume that*

(E^+) *for all $A \in \mathcal{A}$ and all $t \in T$, there is $z \in Z_t$ such that $A \cup \{z\} \in \mathcal{A}$.*

Then there exists $B \in \mathcal{A}$ such that $B \cap Z_t \neq \emptyset$ for all $t \in T$.

We can see the collection $\mathcal{Z}$ as a generalization of the unary operation $'$ from the statement of RTT, since one can put $\mathcal{Z} = \{\{x, x'\} : x \in X\}$. So RTT trivially follows from RTT^+:

Lemma 4.8. *RTT^+ implies RTT.*

Proof. Suppose that RTT^+ holds. Assume the hypothesis of RTT. Then, by putting $\mathcal{Z} = \{\{x, x'\} : x \in X\}$, ($\mathrm{E}^+$) is satisfied. So, by RTT^+ we obtain a desired element $B \in \mathcal{A}$; B satisfies the condition that for all $x \in X$, $x \in B$ or $x' \in B$. □

We further generalize the hypothesis of RTT^+ and obtain the statement of RTT^{++}.

RTT^{++}. *Let X be a set, $\mathcal{A}$ a non-empty system of finite character on X and $\mathcal{Z} = \{Z_t : t \in T\}$ a collection of finite subsets of X indexed by some set T. Assume that*

(E^{++}) *for each finite subset $\{t_1, \dots, t_n\}$ of T, there is a set $\{z_1, \dots, z_n\} \in \mathcal{A}$ such that $z_k \in Z_{t_k}$ for $1 \leq k \leq n$.*

Then there exists $B \in \mathcal{A}$ such that $B \cap Z_t \neq \emptyset$ for all $t \in T$.

Note that also RTT^+ and RTT^{++} can be *enriched* in the same manner as we enriched RTT and other non-constructive principles. One can also prove the equivalence between the enriched form and the plain form for RTT^+ and RTT^{++}, analogously to Theorem 4.1.

Now, the proof of the following lemma will exhibit how (E^{++}) generalizes (E^+).

Lemma 4.9. *RTT^{++} implies RTT^{+}.*

Proof. Suppose that RTT^{++} holds. Assume the hypothesis of RTT^{+}. We check that (E^{++}) is satisfied. Let $\{t_1, \ldots, t_n\}$ be a finite subset of T. Since $\mathcal{A}$ is non-empty and has finite character, $\emptyset \in \mathcal{A}$. So, by (E^{+}), there exists $z_1 \in Z_{t_1}$ s.t. $\{z_1\} = \emptyset \cup \{z_1\} \in \mathcal{A}$. In the same manner we find $z_2 \in Z_{t_2}, \ldots, z_n \in Z_{t_n}$ s.t. $\{z_1, \ldots, z_n\} \in \mathcal{A}$. Therefore (E^{++}) is satisfied and we may invoke $\mathrm{RTT}^{(} + +)$ to obtain a desired element $B \in \mathcal{A}$ satisfying $B \cap Z_t \neq \emptyset$ for all $t \in T$. □

Then it remains to prove that PIT implies RTT^{++}. As announced before, we will prove that the Ultrafilter Lemma implies RTT, which does the same because PIT and UL are equivalent.

Theorem 4.10. *The Ultrafilter Lemma implies RTT^{++}.*

Proof. We prove RTT^{++} using the Ultrafilter Lemma.

Let $\mathcal{A}$ be a non-empty system of finite character on a set X and $\mathcal{Z} = \{Z_t : t \in T\}$ a collection of finite subsets of X indexed by a set T that satisfies (E^{++}). Define

$$H = \prod_{t \in T} Z_t$$

and choose $g \in H$ using the Axiom of Choice for Finite Sets, which by Theorem 3.15 follows from the Ultrafilter Lemma. For each $F \Subset T$, define

$$H_F = \{f \in H : f(F) \in \mathcal{A}\}$$

and let

$$\mathcal{H} = \{H_F : F \Subset T\}.$$

First we prove (a) that $H_F \neq \emptyset$ for all $F \Subset T$. By (E^{++}) there exists a function ϕ' on F such that $\phi'(t) \in Z_t$ for all $t \in F$ and $\phi'(F) \in \mathcal{A}$. Now, define the function ϕ on T by

$$\phi(t) = \begin{cases} \phi'(t) & \text{if } t \in F, \\ g(t) & \text{otherwise.} \end{cases}$$

Then $\phi \in H$ and $\phi(F) = \phi'(F) \in \mathcal{A}$. So $\phi \in H_F$, as desired.

Then we prove (b) that $H_{F \cup G} \subseteq H_F \cap H_G$ for all $F, G \Subset T$. Let $\phi \in H_{F \cup G}$. Then $\phi(F \cup G) \in \mathcal{A}$. Since $\phi(F), \phi(G) \Subset \phi(F \cup G)$, by the finite character of $\mathcal{A}$, $\phi(F), \phi(G) \in \mathcal{A}$. Therefore $\phi \in H_F \cap H_G$, as desired.

From (a) and (b) it follows that the non-empty collection $\mathcal{H}$ has the finite intersection property, because if $F_1, \ldots, F_n \Subset T$ then $H_{F_1} \cap \cdots \cap H_{F_n} \supseteq H_{F_1 \cup \cdots \cup F_n} \neq \emptyset$. So by Lemma 3.13 and the Ultrafilter Lemma, we obtain an ultrafilter $\mathcal{U}$ of $\mathcal{P}(H)$ such that $\mathcal{H} \subseteq \mathcal{U}$.

We claim (c) that for each $t \in T$, there exists a unique $z_t \in Z_t$ such that $H_t \in \mathcal{U}$, where

$$H_t := \{f \in H : f(t) = z_t\}.$$

Let $t \in T$ and write $Z_t = \{z_1, \dots, z_n\}$. For each integer k, $1 \leq k \leq n$, define

$$H_k = \{f \in H : f(t) = z_k\}.$$

Then clearly $H_1 \cup \cdots \cup H_n = H$. We also have that $H_k \cap H_{k'} = \emptyset$ for $1 \leq k < k' \leq n$, because if $f \in H_k$ then $f(t) = z_k \neq z_{k'}$ and so $\phi \notin H_{k'}$. By Lemma 3.14, there is a unique integer $k(t)$, $1 \leq k(t) \leq n$, such that $H_k(t) \in \mathcal{U}$. Putting $z_t = z_{k(t)}$, we have that $H_t = H_{k(t)} \in \mathcal{U}$ and that (c) is proved.

Now, define

$$\mathcal{B} = \{z_t : t \in T\}.$$

Then $B \cap Z_t \neq \emptyset$ for all $t \in T$.

To prove that $B \in \mathcal{A}$, we show that all finite subsets of B are in $\mathcal{A}$. Let $\{z_{t_1}, \dots, z_{t_n}\}$ be a finite subset of B and write $F = \{t_1, \dots, t_n\}$. Since $F \Subset T$, $H_F \in \mathcal{H} \subseteq \mathcal{U}$. And by (c), $H_{t_1}, \dots, H_{t_n} \in \mathcal{U}$. Thus, by the finite intersection property of $\mathcal{U}$, there exists $f \in H$ such that

$$f \in H_F \cap H_{t_1} \cap \cdots \cap H_{t_n}.$$

Since $f(t_k) = z_{t_k}$ for all $1 \leq k \leq n$ and $f \in H_F$, we have that

$$\{z_{t_1}, \dots, z_{t_n}\} = f(F) \in \mathcal{A},$$

as desired. By the finite character of $\mathcal{A}$, we conclude that $B \in \mathcal{A}$.

We found a desired element $B \in \mathcal{A}$ such that $B \cap Z_t \neq \emptyset$ for all $t \in T$. This proves that the Ultrafilter Lemma implies RTT^{++}. □

Corollary 4.11. *The following are equivalent.*

(1) RTT
(2) RTT^{+}
(3) RTT^{++}
(4) PIT

Proof. We have just seen in Theorem 4.10 that UL implies RTT^{++}. Since PIT and UL are equivalent by Lemma 3.12, we have that (4)⇒(3). (3)⇒(2) is Lemma 4.9, (2)⇒(1) is Lemma 4.8 and (1)⇒(4) is Theorem 4.2. Therefore (1)⇔(2)⇔(3)⇔(4). □

4.3 Applications of RTT^{++}

Now we turn to the applications of RTT^{++}. We will observe various principles that are naturally equivalent to or easily follow from RTT^{++}. These are all introduced in Hodel's original paper [1]. We will provide detailed proofs for the equivalences and the implications.

The first application we will consider is the Cowen-Engeler Lemma:

Cowen-Engeler Lemma. *Let T and X be sets. Let $\mathcal{E}$ be a collection of functions from subsets of T into X such that*
(a) for all $t \in T$, the set $X_t = \{\phi(t) : \phi \in \mathcal{E} \wedge t \in dom(\phi)\}$ is finite,
(b) for all $F \Subset T$, there is a function $\phi \in \mathcal{E}$ whose domain is F,
(c) $\mathcal{E}$ has finite character (that is, a function ϕ from a subset T into X is in $\mathcal{E}$ if and only if for every finite $F \subset dom(\phi)$, $\phi|_F$ is in $\mathcal{E}$.
Then T is the domain of some $\phi \in \mathcal{E}$.

We will give a direct proof that the Cowen-Engeler Lemma is equivalent to RTT^{++}.

Lemma 4.12. *RTT^{++} implies the Cowen-Engeler Lemma.*

Proof. We prove that Cowen-Engeler Lemma follows from RTT^{++}. Let T, X and $\mathcal{E}$ as in the hypothesis of the Cowen-Engeler Lemma. For each $t \in T$, define $Z_t = \{(t, x) : x \in X_t\}$.

Now, to apply RTT^{++}, we check (E^{++}). Let $F = \{t_1, \dots, t_n\} \subseteq T$. By (b), there is a function $\phi \in \mathcal{E}$ whose domain is F. We now have that

$$\phi = \{(t_1, \phi(t_1)), \dots, (t_n, \phi(t_n))\} \in \mathcal{E} \text{ and } (t_k, \phi(t_k)) \in Z_{t_k} \text{ for } 1 \leq k \leq n.$$

This proves that (E^{++}) is satisfied. Now we obtain, by RTT^{+}, a function $\varphi \in \mathcal{E}$ such that $\varphi \cap Z_t \neq \emptyset$. So the function φ is defined on all of T, as desired. This proves that the Cowen-Engeler Lemma follows from RTT^{++}. □

Lemma 4.13. *The Cowen-Engeler Lemma implies RTT^{++}*

Proof. We prove that the Cowen-Engeler Lemma implies RTT^{++}. Let $\mathcal{A}$ be a non-empty system on X with finite character and let $\{Z_t : t \in T\}$ be a collection of finite subsets of X such that the condition (E^{++}) is satisfied. Define

$$\begin{aligned} \mathcal{E} \quad = \quad & \{\phi : \phi \text{ is a function with } \text{dom}(\phi) \subseteq T, \phi(t) \in Z_t \text{ for all } t \in \text{dom}(\phi) \\ & \text{and for all } F \Subset \text{dom}(\phi), \{\phi(t) : t \in F\} \in \mathcal{A}\}. \end{aligned}$$

First we check (a) that $\mathcal{E}$ has finite character. Let $S \subseteq T$ and $\phi : S \to X$. If $\phi \in \mathcal{E}$, then by the definition of $\mathcal{E}$, $\phi|_F \in \mathcal{E}$ for all $F \Subset S$. Now, suppose that ϕ is such that $\phi|_F \in \mathcal{E}$ for all $F \Subset S$. Then (1) $\text{dom}(\phi) = S \subseteq T$, (2) if $t \in \text{dom}(\phi)$ then $\phi(t) = \phi|_{\{t\}}(t) \in Z_t$ and (3) for every $F \Subset \text{dom}(\phi)$, $\{\phi(t) : t \in F\} \in \mathcal{A}$. Therfore $\phi \in \mathcal{E}$. So (a) is satisfied.

Now we check (b). Let $F \Subset T$. Then by (E^{++}), there is a function $\phi : F \to X$ s.t. $\phi(t) \in Z_t$ for all $t \in F$. And if $F' \Subset F$ then $\phi(F') \in \mathcal{A}$ because $\phi(F') \Subset \phi(F) \in \mathcal{A}$ and $\mathcal{A}$ has finite character. So $\phi \in \mathcal{E}$, as desired.

Lastly, we check (c). Let $t \in T$. Note that $X_t = \{\phi(t) : \phi \in \mathcal{E} \wedge t \in \mathrm{dom}(\phi)\} \subseteq Z_t$. Since Z_t is finite, X_t is finite.

Now the hypothesis of the Cowen-Engeler Lemma is satisfied, so we obtain a function $\phi \in \mathcal{E}$ with domain T. Let $B = \{\phi(t) : t \in T\}$. Clearly $B \cap Z_t \neq \emptyset$. To prove $B \in \mathcal{A}$, we use the finite character of $\mathcal{A}$. Let $B' \Subset B$. Then $B' = \phi(F)$ for some $F \Subset T$. Now, $B' = \phi(F) \in \mathcal{A}$ holds because $\phi \in \mathcal{E}$. So $B \in \mathcal{A}$. B is a desired element to conclude RTT^{++}. □

Corollary 4.14. *RTT^{++} is equivalent to the Cowen-Engeler Lemma.*

Proof. This is proved by Lemma 4.12 and 4.13. □

Now we look at another application of RTT^{++}.

Generalized Consistency Theorem. *Let $\{X_t : t \in T\}$ be a collection of finite non-empty sets, and let $\mathcal{E}$ be a collection of functions ϕ such that dom(ϕ) is a finite subset of T and $\phi(t) \in X_t$ for all $t \in$ dom(ϕ), i.e. ϕ is a finite choice function for $\{X_t : t \in T\}$. Assume that*

(1) for each $F \Subset T$, there exists $\phi \in \mathcal{E}$ such that dom(ϕ) $= F$,
(2) if $\phi \in \mathcal{E}$ and $F \subseteq$ dom(ϕ) then $\phi|_F \in \mathcal{E}$.

Then there is a choice function Φ for $\{X_t : t \in T\}$ such that $\Phi|_F \in \mathcal{E}$ for all $F \Subset T$.

This statement is a generalized version of the Consistency Theorem from [5]. In the original statement, the codomain of each function ϕ is $\{0, 1\}$.

For proving the Generalized Consistency Theorem with RTT^{++}, we first do the following lemma. It facilitates constructing a collection of finite character from a given collection of functions.

Lemma 4.15. *Let T and X be non-empty sets and let $\mathcal{E}$ be a collection of functions from finite subsets of T into X such that for all $\phi \in \mathcal{E}$ and all $F \subseteq$ dom(ϕ), $\phi|_F \in \mathcal{E}$. Let*

$$\begin{aligned} \mathcal{A} = \{&\phi : \phi \textit{ is a function from a subset of } T \textit{ into } X \\ &\textit{and for all finite } F \subseteq dom(\phi),\ \phi|_F \in \mathcal{E}\}. \end{aligned}$$

Then

(1) $\mathcal{E} \subseteq \mathcal{A}$,
(2) if $\phi \in \mathcal{A}$ and dom(ϕ) is finite, then $\phi \in \mathcal{E}$,
(3) $\mathcal{A}$ has finite character.

Proof. First we check (1). Let $\phi \in \mathcal{E}$. Then ϕ is a function from a (even finite) subset of T into X such that for all $F \Subset \mathrm{dom}(\phi)$, $\phi|_F \in \mathcal{E}$. Thus $\phi \in \mathcal{A}$.

Then we check (2). Let $\phi \in \mathcal{A}$ such that $\mathrm{dom}(\phi)$ is finite. Then for all $F \subseteq \mathrm{dom}(\phi)$, which is thus finite, $\phi|_F \in \mathcal{E}$. Therefore $\phi \in \mathcal{E}$.

Now we check (3), i.e. for all function ϕ on a subset S of T into X, $\phi \in \mathcal{A}$ if and only if all finite subfunctions of ϕ are in $\mathcal{A}$. 'only if': Suppose $\phi \in \mathcal{A}$. Let $F \Subset S$ and consider the finite subfunction $\phi|_F$. If $\phi|_G$ is a finite subfunction of $\phi|_F$ for $G \Subset F$, then $\phi|_G \Subset \phi$, so $\phi|_G \in \mathcal{A}$ by the finite character of $\mathcal{A}$. Thus $\phi|_F \in \mathcal{A}$, as desired. For 'if', we prove the contrapositive. Suppose that $\phi \notin \mathcal{A}$. Then, by definition of $\mathcal{A}$, there exist $F \Subset S$ such that $\phi|_F \notin \mathcal{E}$. Note that $\phi|_F$ is a finite subfunction of itself such that $\phi|_F \notin \mathcal{E}$. So, by definition of $\mathcal{A}$, $\phi|_F \notin \mathcal{A}$. Thus there exists a finite subfunction $\phi|_F \notin \mathcal{A}$ of ϕ, as desired. This proves (3) that $\mathcal{A}$ has finite character.

We conclude that (1), (2) and (3) hold. □

Lemma 4.16. *RTT^{++} implies the Generalized Consistency Theorem.*

Proof. Define

$$\begin{aligned}\mathcal{B} \;=\; & \{\phi : \phi \text{ is a function with } \mathrm{dom}(\phi) \subseteq T,\ \phi(t) \in X_t \text{ for all} \\ & t \in \mathrm{dom}(\phi), \text{ and for all finite } F \subseteq \mathrm{dom}(\phi),\ \phi|_F \in \mathcal{E}\}.\end{aligned}$$

Let $\mathcal{A}$ be constructed from $\mathcal{E}$ as in Lemma 4.15. We prove that $\mathcal{A} = \mathcal{B}$. Clearly $\mathcal{B} \subseteq \mathcal{A}$. To show that $\mathcal{A} \subseteq \mathcal{B}$, let $\phi \in \mathcal{A}$. It suffices to prove that $\phi(t) \in X_t$ for all $t \in \mathrm{dom}(\phi)$. This is indeed the case, because $\phi(t) = \phi|_{\{t\}}(t) \in X_t$ for all $t \in \mathrm{dom}(\phi)$. This proves that $\mathcal{A} = \mathcal{B}$.

Now from Lemma 4.15 it follows that by (a), $\mathcal{E} \subseteq \mathcal{B}$, and by (c), $\mathcal{B}$ has finite character.

For each $t \in T$, define

$$Z_t = \{(t, x) : x \in X_t\}.$$

To apply RTT^{++} on $\mathcal{B}$, we check (E^{++}). Let $F = \{t_1, \ldots, t_n\} \subseteq T$. By (1), there exists $\phi \in \mathcal{E}$ such that $\mathrm{dom}(\phi) = \mathcal{F}$. Then, since $\mathcal{E} \subseteq \mathcal{B}$, $\phi \in \mathcal{B}$. And since

$$\phi = \{(t_1, \phi(t_1)), \ldots, (t_n, \phi(t_n))\},$$

we have that $(t_k, \phi(t_k)) \in Z_{t_k}$ for each $1 \leq k \leq n$, as desired by (E^{++}).

By RTT^{++}, we find $\Phi \in \mathcal{A}$ such that $\Phi \cap Z_t \neq \emptyset$ for all $t \in T$. This means that $\mathrm{dom}(\Phi) = T$, as desired.

We conclude that the Generalized Consistency Theorem follows from RTT^{++}. □

The converse implication also naturally holds. Here is a proof.

Lemma 4.17. *The Generalized Consistency Theorem implies RTT^{++}.*

Proof. Let $\mathcal{A}$ be a non-empty system with finite character on a set X and $\mathcal{Z} = \{Z_t : t \in T\}$ be a collection of finite subsets of X s.t. (E^{++}) is satisfied. Define

$$\begin{aligned}\mathcal{E} &= \{\phi : \phi \text{ is a function on a finite subset of } T \\ &\quad \text{and } \phi(t) \in Z_t \text{ for all } t \in \text{dom}(\phi)\}.\end{aligned}$$

From (E^{++}) it follows that for each $F \Subset T$, there is $\phi \in \mathcal{E}$ such that $\text{dom}(\phi) = F$. And by definition of $\mathcal{E}$, if $\phi \in \mathcal{E}$ and $F \subseteq \text{dom}(\phi)$ then $\phi|_F \in \mathcal{E}$. So, by the Generalized Consistency Theorem, we obtain a choice function Φ for $\mathcal{Z}$ such that $\Phi|_F \in \mathcal{E}$ for all $F \Subset T$.

Let $B = \Phi(T)$. We claim that $B \cap Z_t \neq \emptyset$ for all $t \in T$. Let $t \in T$. Then $\Phi(t) \in \Phi(T) \cap Z_t = B \cap Z_t$, as desired. So B is a desired element that we sought.

This proves that the Generalized Consistency Theorem implies RTT^{++}. □

Corollary 4.18. *RTT^{++} and the Generalized Consistency Theorem are equivalent.*

Proof. This is exactly Lemma 4.16 and Lemma 4.17. □

We consider the following Selection Lemma due to Rado [6], which easily follows from the Generalized Consistency Theorem, as another application of RTT^{++}.

Rado's Selection Lemma. *Let $\{X_t : t \in T\}$ be a collection of finite non-empty sets. Assume that for each $B \Subset T$, there is a choice function ϕ_B for $\{X_t : t \in B\}$ (the domain of ϕ_B is B and $\phi(t) \in X_t$ for all $t \in T$). Then there is a choice function Φ for $\{X_t : t \in B\}$ such that for all $F \Subset T$, there is a $B \Subset T$ such that $F \subseteq B$ and $\Phi|_F = \phi_B|_F$.*

Proposition 4.19. *The Generalized Consistency Theorem implies Rado's Selection Lemma.*

Proof. Define

$$\mathcal{E} = \{\phi_B|_F : F \subseteq B, B \Subset T\}.$$

Since for all $F \Subset T$, $\phi_F \in \mathcal{E}$, $\mathcal{E}$ satisfies (1) in the hypothesis of the Generalized Consistency Theorem. And if $\phi \in \mathcal{E}$ and $F \subseteq \text{dom}(\phi)$, then $\phi|_F \in \mathcal{E}$ by definition of $\mathcal{E}$. So $\mathcal{E}$ also satisfies (2).

By the Generalized Consistency Theorem, we obtain a choice function Φ for $\{X_t : t \in T\}$ such that $\phi|_F \in \mathcal{E}$ for all $F \Subset T$. This means that $\phi|_F = \phi_B|_F$ for some $B \Subset T$ with $F \subseteq B$, as desired.

This proves that the Generalized Consistency Theorem implies Rado's Selection Lemma. □

Remark 4.20. Unlike the other two application, Cowen-Engeler Lemma and Generalized Consistency Theorem, that we have seen, Rado's Selection Lemma is not equivalent to RTT^{++} and hence to PIT. This means that the Selection Lemma does not imply PIT in ZF. See [7] for a proof of this fact.

5 Another restriction of TT

In this section we discuss another restriction of TT: the *Finite Cutset Lemma*. This restriction is due to M. Erné [2]. In his paper, the equivalence between PIT and the Finite Cutset Lemma is proved via Alexander's Subbase Theorem and another principle for systems of finite character named *Intersection Lemma*. Instead of this approach, we will directly provide an equivalence proof between the Finite Cutset Lemma and RTT$^+$. Thereafter we will give the one-direction proofs that the Alexander's Subbase Theorem implies the Intersection Lemma and the Intersection Lemma implies the Finite Cutset Lemma. Since in Section 4 we already proved that RTT (and hence RTT$^+$) implies Alexander's Subbase Theorem, this will establish the equivalences between all of these principles.

5.1 Finite Cutset Lemma

First, we study the statement of the Finite Cutset Lemma. We will see how the restriction is made from the Tukey-Teichmüller Theorem this time, and discuss the difference to RTT. We begin by introducing some terminologies.

Definition 5.1. Given sets A and B we say that A and B *intersect*, or A *intersects* B, if $A \cap B \neq \emptyset$.

Let $\mathcal{A}$ be a system on a set X. A set $C \subseteq X$ is called a *cutset* of $\mathcal{A}$ if for all $A \in \mathcal{A}$ there is an extension $A' \supseteq A$ in $\mathcal{A}$ such that A' intersects C.

The best way to describe the relation between the Finite Cutset Lemma and the Tukey-Teichmüller Theorem, I think, is to begin with the following sibling principle of the Finite Cutset Lemma, which is equivalent to TT.

Cutset Lemma. *Let $\mathcal{A}$ be a non-empty system of finite character on a set X. Then there is $M \in \mathcal{A}$ such that M intersects every cutset of $\mathcal{A}$.*

Note that the hypothesis of the Cutset Lemma is the same as that of the Tukey-Teichmüller Theorem. So if we have that the conclusions of the Cutset Lemma and of the TT are equivalent, the two principles must be equivalent. The next lemma will prove that the element returned by the Cutset Lemma is indeed the maximal element of the system.

Lemma 5.2. *If a system $\mathcal{A}$ on a set X has finite character, then the following two conditions on $M \in \mathcal{A}$ are equivalent:*

(1) M is maximal,
(2) M intersects every cutset of $\mathcal{A}$.

Proof. First we prove that (1) implies (2). Suppose that M is maximal. Let $C \subseteq X$ be a cutset of $\mathcal{A}$. Then there exist $M' \in \mathcal{A}$ with $M \subseteq M'$ such that C intersects M'. But by the maximality of M, $M' = M$. Therefore C intersects M, as desired. So M intersects every cutset of $\mathcal{A}$.

For the part that (2) implies (1), we prove the contrapositive. Suppose that M is not maximal. Then there exists $M' \in \mathcal{A}$ such that $M \subsetneq M'$.

We claim that $M^c := X - M$ is a cutset of $\mathcal{A}$. Let $A \in \mathcal{A}$. Consider the case $A \subseteq M'$. Then M^c intersects $M' \supseteq A$, because $M \subsetneq M'$. Consider the case $A \nsubseteq M'$. Then there exists $x \in A$ such that $x \notin M'$ and so $x \notin M$. Thus M^c intersects A. This proves that M^c is a cutset of $\mathcal{A}$.

But M^c is disjoint from M and so M^c does not intersect M, as desired to prove that not every cutset intersects M.

We conclude that (1) and (2) are equivalent. □

Corollary 5.3. *The Cutset Lemma and the Tukey-Teichmüller Theorem are equivalent.*

Proof. The hypotheses of the two principles are exactly the same. And by Lemma 5.2, also the conclusions are equivalent. Therefore the Cutset Lemma and the Tukey-Teichmüller Theorem must be equivalent. □

We just add one word *finite* to the conclusion of the Cutset Lemma, and obtain the *Finite Cutset Lemma*.

Finite Cutset Lemma. *Let $\mathcal{A}$ be a non-empty system of finite character on a set X. Then there is $M \in \mathcal{A}$ such that M intersects every* finite *cutset of $\mathcal{A}$.*

Note that since we can view the TT and the Cutset Lemma as the same thing by the previous discussions, we may indeed consider the Finite Cutset Lemma as a restriction of the Tukey-Teichmüller Theorem. We observe that the restriction is simpler than RTT in the sense that while RTT changed both the hypothesis and the conclusion, the Finite Cutset Lemma only weakened the conclusion. So the types of the two restrictions are different, and for this reason one may worry that a direct equivalence proof between them would be very tricky. But we will see that, fortunately, the conditions of the Finite Cutset Lemma match those of RTT very well.

As announced before, RTT^+ is the variation of RTT that we use to prove the equivalence with the Finite Cutset Lemma. First we derive the Finite Cutset Lemma from RTT^+. The main of idea of this proof is that we can take the collection of all finite cutsets as $\mathcal{Z}$ in the hypothesis of RTT^+.

Lemma 5.4. *RTT^+ implies the Finite Cutset Lemma.*

Proof. Let X be a set and $\mathcal{A}$ be a non-empty system of finite character on X. Define $\mathcal{Z} = \{Z \subseteq X : Z \text{ is a finite cutset of } \mathcal{A}\}$.

We check that $\mathcal{A}$ satisfies the extension property (E^+) w.r.t. $\mathcal{Z}$. Let $A \in \mathcal{A}$ and $Z \in \mathcal{Z}$. Since Z is a cutset for $\mathcal{A}$, there is $A' \supseteq A$ in $\mathcal{A}$ such that $A' \cap Z \neq \emptyset$. Let $c \in A' \cap Z \subseteq Z$. We verify $A \cup \{c\} \in \mathcal{A}$ as follows. Let F be a finite subset of $A \cup \{c\}$. Then F is a finite subset of $A' \in \mathcal{A}$, so by the finite character of $\mathcal{A}$, $F \in \mathcal{A}$. Thus, again by the finite character of $\mathcal{A}$, $A \cup \{c\} \in \mathcal{A}$, as desired. Therefore (E^+) is satisfied.

Now, by RTT^+ we acquire an element $B \in \mathcal{A}$ such that $B \cap Z_t \neq \emptyset$. In other words, B intersects every finite cutset of $\mathcal{A}$. This proves that Finite Cutset Lemma follows from RTT^+. □

Also in the proof of the converse, which we now give, the connection between the weakened conclusion in the Finite Cutset Lemma and the conditions of RTT is very explicit. Each element of $\mathcal{Z}$ in the hypothesis of RTT^+, will turn out, because of (E^+), to be a cutset. So in particular, an element that intersects every finite cutset, intersects every element of $\mathcal{Z}$.

Lemma 5.5. *The Finite Cutset Lemma implies RTT^+.*

Proof. Let X be a set, $\mathcal{A}$ be a non-empty system of finite character on X and $\mathcal{Z} = \{Z_t : t \in T\}$ be a collection of finite subsets of X indexed by a set T such that (E^+) is satisfied.

As anticipated, we will see that each $Z_t \in \mathcal{Z}$ is a cutset. Let $t \in T$. We check that Z_t is a cutset for $\mathcal{A}$. Let $A \in \mathcal{A}$. By (E^+), there exists $z \in Z_t$ such that $A \cup \{z\} \in \mathcal{A}$. In other words, $A \cup \{z\} \supseteq A$ intersects Z_t. Therefore Z_t is a cutset, as desired.

Now, since $\mathcal{A}$ has finite character, the Finite Cutset Lemma gives an element B of $\mathcal{A}$ that intersects every finite cutset for $\mathcal{A}$. In particular, $B \cap Z_t \neq \emptyset$ for all $t \in T$. This proves that RTT^+ follows from the Finite Cutset Lemma. □

Corollary 5.6. *RTT is equivalent to FCL.*

Proof. By Lemma 5.4 and 5.5, RTT^+ is equivalent to FCL. Since RTT is equivalent to RTT^+, also RTT is equivalent to FCL. □

Before we turn to the discussion of the Intersection Lemma, we state the enriched form of the Finite Cutset Lemma.

Finite Cutset Lemma (enriched form). *Let $\mathcal{A}$ be a system of finite character on a set X. Then for each $A \in \mathcal{A}$, there is $M \in \mathcal{A}$ such that $A \subseteq M$ and M intersects every finite cutset of $\mathcal{A}$.*

Remark 5.7. Since we have seen enough proofs of the equivalence between the plain and enriched forms, and we will not need the enriched form of FC in the further discussions, we will not state the equivalence proof this time. But for the matter of fact, we surely know that the plain and enriched forms of the Finite Cutset Lemma are equivalent, because Erné [2] proves that the enriched form is equivalent to PIT, which in turn is equivalent to the plain form by our discussions above.

5.2 Intersection Lemma

Intersection Lemma. *If a system $\mathcal{A}$ on a set X has finite character, then so does the system*

$$\{\mathcal{F} \subseteq \mathcal{P}_{<\omega}(X) : \exists S \in \mathcal{A} \forall F \in \mathcal{F}(S \cap F \neq \emptyset)\},$$

i.e. the system of all collections of finite subsets of X intersecting a common member of $\mathcal{A}$.

Like the the Finite Cutset Lemma, also the Intersection Lemma is from [2]. As noticed before, we will use this new principle for systems of finite character just stated, to bridge the implication that Alexander's Subbase Theorem implies the Finite Cutset Lemma. This implication will finish the proof of equivalences promised in Theorem 1.2.

First we observe how Alexander's Subbase Theorem proves the Intersection Lemma.

Lemma 5.8. *Alexander's Subbase Theorem implies the Intersection Lemma.*

Proof. Let $\mathcal{A}$ be a system of finite character on a set X. For each $x \in X$, define

$$\mathcal{A}_x = \{S \in \mathcal{A} : x \notin S\}$$

and let $\mathscr{T}$ be the topology on $\mathcal{A}$ generated by the subbase

$$\mathscr{B} = \{\mathcal{A}_x : x \in X\}.$$

We claim (I) that if $Y \subseteq X$, then $\mathcal{A} \neq \bigcup\{\mathcal{A}_x : x \in Y\} \Leftrightarrow Y \in \mathcal{A}$. '$\Rightarrow$': Suppose $\mathcal{A} \neq \bigcup\{\mathcal{A}_x : x \in Y\}$. Since $\mathcal{A} \supseteq \bigcup\{\mathcal{A}_x : x \in Y\}$, this means that there is $S \in \mathcal{A}$ s.t. $S \notin \bigcup\{\mathcal{A}_x : x \in Y\}$. Thus for all $x \in Y$, $S \notin \mathcal{A}_x$ and so $x \in S$. This means $Y \subseteq S$. Now, by the finite character of $\mathcal{A}$, since all finite subsets of Y are finite subsets of S, we conclude $Y \in \mathcal{A}$. '$\Leftarrow$': Suppose $Y \in \mathcal{A}$. Then for all $x \in Y$, $Y \notin \mathcal{A}_x$. So $Y \notin \bigcup\{\mathcal{A}_x : x \in Y\}$ while $Y \in \mathcal{A}$, which proves that $\mathcal{A} \neq \bigcup\{\mathcal{A}_x : x \in Y\}$ as desired.

Now we check that the space $(\mathcal{A}, \mathscr{T})$ satisfies the hypothesis of Alexander's Subbase Theorem. Let $\{\mathcal{A}_x : x \in Y\} \subseteq \mathscr{B}$ be a cover for $\mathcal{A}$ indexed by $Y \subseteq X$. By (I), $Y \notin \mathcal{A}$. Thus, by the finite character of $\mathcal{A}$, there is $F \Subset Y$

such that $F \notin \mathcal{A}$. We again use (I) and find that $\{\mathcal{A}_x : x \in F\}$ is a subcover of $\{\mathcal{A}_x : x \in Y\}$, as desired. So by the Alexander's Subbase Theorem, the space $(\mathcal{A}, \mathscr{T})$ is compact.

For each $F \Subset X$, define

$$\mathcal{A}_F = \{S \in \mathcal{A} : F \cap S = \emptyset\}.$$

Note that $S \in \mathcal{A}_F \Leftrightarrow F \cap S = \emptyset \Leftrightarrow \forall x \in F(x \notin S) \Leftrightarrow \forall x \in F(S \in \mathcal{A}_x) \Leftrightarrow S \in \bigcap\{\mathcal{A}_x : x \in F\}$. Therefore we have that $\mathcal{A}_F = \bigcap\{\mathcal{A}_x : x \in F\} \in \mathscr{T}$ because F is finite.

Now we claim (II) that if $\mathcal{F} \subseteq \mathcal{P}_{<\omega}(X)$, then $\mathcal{A} \neq \bigcup\{\mathcal{A}_F : F \in \mathcal{F}\} \Leftrightarrow \exists S \in \mathcal{A} \forall F \in \mathcal{F}(S \cap F \neq \emptyset)$. '$\Rightarrow$': Suppose $\mathcal{A} \neq \bigcup\{\mathcal{A}_F : F \in \mathcal{F}\}$. Since $\mathcal{A} \supseteq \bigcup\{\mathcal{A}_F : F \in \mathcal{F}\}$, this means that there is $S \in \mathcal{A}$ s.t. $S \notin \bigcup\{\mathcal{A}_F : F \in \mathcal{F}\}$. Thus for all $F \in \mathcal{F}$, $S \notin \mathcal{A}_F$ and so $F \cap S \neq \emptyset$, as desired. '$\Leftarrow$': Suppose that $\exists S \in \mathcal{A} \forall F \in \mathcal{F}(S \cap F \neq \emptyset)$. Then $S \notin \mathcal{A}_F$ for all $F \in \mathcal{F}$. So $S \notin \bigcup\{\mathcal{A}_F : F \in \mathcal{F}\}$ while $S \in \mathcal{A}$, which proves that $\mathcal{A} \neq \bigcup\{\mathcal{A}_F : F \in \mathcal{F}\}$ as desired.

Finally we verify that the system

$$\mathscr{F} = \{\mathcal{F} \subseteq \mathcal{P}_{<\omega}(X) : \exists S \in \mathcal{A} \forall F \in \mathcal{F}(S \cap F \neq \emptyset)\}$$

has finite character. Let $\mathcal{F} \subseteq \mathcal{P}_{<\omega}(X)$. Suppose that $\mathcal{F} \in \mathscr{F}$. Then there is $S \in \mathcal{A}$ such that $S \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Thus if $\mathcal{G}$ is a finite subcollection of $\mathcal{F}$, then $S \cap G \neq \emptyset$ for all $G \in \mathcal{G} \subseteq \mathcal{F}$, so $\mathcal{G} \in \mathscr{F}$ as desired. For the converse, suppose that all finite subcollections of $\mathcal{F}$ are in $\mathscr{F}$. Suppose, for contradiction, that $F \notin \mathscr{F}$. Then by (II), $\{\mathcal{A}_F : F \in \mathcal{F}\}$ covers $\mathcal{A}$. By the compactness of $(\mathcal{A}, \mathscr{T})$, there is $\mathcal{G} \Subset \mathcal{F}$ such that $\{\mathcal{A}_F : F \in \mathcal{G}\}$ is a finite subcover of $\{\mathcal{A}_F : F \in \mathcal{F}\}$. Now, again by (II), we find that $\mathcal{G} \notin \mathscr{F}$, which contradicts $\mathcal{G}$ being a finite subset of $\mathcal{F}$. Therefore $\mathcal{F} \in \mathscr{F}$. This proves that $\mathscr{F}$ has finite character.

We conclude that Alexander's Subbase Theorem implies the Intersection Lemma. □

Then we derive the Finite Cutset Lemma from the Intersection Lemma.

Lemma 5.9. *The Intersection Lemma implies the Finite Cutset Lemma.*

Proof. Let $\mathcal{A}$ be a non-empty system of finite character on a set X. And let $\mathscr{F}$ be the system of all collections of finite subsets of X intersecting a common member of $\mathcal{A}$. Define

$$\mathcal{F} = \{F \Subset X : \forall A \in \mathcal{A} \exists x \in \mathcal{F}(S \cup \{x\} \in \mathcal{A})\}.$$

To prove that $\mathcal{F} \in \mathscr{F}$, we show that all finite subcollections of $\mathcal{F}$ are in $\mathscr{F}$. Let $\mathcal{G} \Subset \mathcal{F}$. Since all sets in $\mathcal{G}$ are finite and $\mathcal{G}$ is finite, $\bigcup \mathcal{G}$ is finite. Therefore the collection $\mathcal{B} = \{A \in \mathcal{A} : A \subseteq \bigcup \mathcal{G}\}$ is finite and has a maximal element S. Let $G \in \mathcal{G}$. Then there is $x \in G$ such that $S \cup \{x\} \in \mathcal{A}$. Since $x \in G \subseteq \bigcup \mathcal{G}$, we have that $S \cup \{x\} \subseteq \bigcup \mathcal{G}$ and so $S \cup \{x\} \in \mathcal{B}$. But S is maximal in $\mathcal{B}$, so $S \cup \{x\} = S$. Thus $x \in G \cap S \neq \emptyset$, which implies that $\mathcal{G} \in \mathscr{F}$. Now, by the finite character of $\mathcal{F}$ due to the Intersection Lemma, it follows that $\mathcal{F} \in \mathscr{F}$.

Let $M \in \mathcal{A}$ such that each $F \in \mathcal{F}$ intersects M. To establish that M intersects every finite cutset for $\mathcal{A}$. Let C be a finite cutset for $\mathcal{A}$. Then for all $A \in \mathcal{A}$, $C \cap A \neq \emptyset$, i.e. there is $x \in C$ such that $x \in A$. Thus $A \cup \{x\} = A \in \mathcal{A}$, which proves that $C \in \mathcal{F}$. Therefore M intersects C, as desired.

We conclude that the Intersection Lemma implies the Finite Cutset Lemma. □

Now, enjoy the following "proof".

Proof of Theorem 1.2. See Figure 1. □

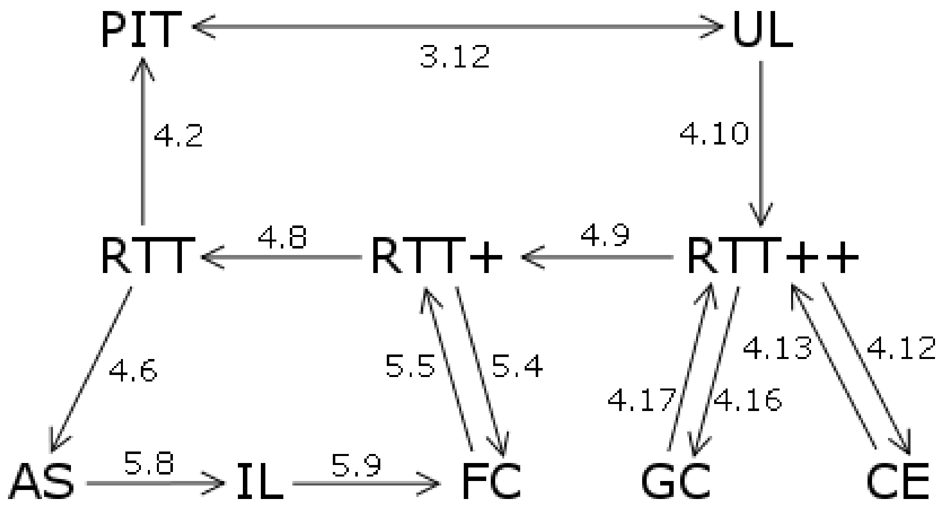

Figure 1: Summary of Theorem 1.2.

6 Applications in Propositional Logic

In this section we will derive Lindenbaum's Theorem, the Model Existence Theorem and the Compactness Theorem in the propositional logic from the Restricted Tukey-Teichmüller Theorem. So from now on, we assume that RTT is valid. First we summarize the basic notions about the propositional logic.

Definition 6.1. We define FOR as the set of all formulas of the propositional logic, with the connectives $\neg$ and $\vee$. A *truth assignment* is a function $\phi : \text{FOR} \to \{T, F\}$ such that

(1) $\phi(A) \neq \phi(\neg A)$,
(2) $\phi(A \vee B) = T$ if and only if $\phi(A) = T$ or $\phi(B) = T$,

for all formulas A and B.

If $\Gamma \subseteq \text{FOR}$, then

- Γ is *satisfiable* if there is at least one truth assignment ϕ such that $\phi(A) = T$ for every $A \in \Gamma$.
- Γ is *finitely satisfiable* if every finite subset of Γ is satisfiable.
- Γ is *consistent* if there is no formula A such that both $\Gamma \vdash A$ and $\Gamma \vdash \neg A$.

The following lemma claims that a system of consistent set of formulas has finite character and extension property. For the extension property proof, we will assume that the Deduction Theorem is provided.

Lemma 6.2. *Let $\Gamma \subset$ FOR. Then*

(C1) Γ *is consistent if and only if every finite subset of* Γ *is consistent.*
(C2) If Γ *is consistent and* A *is any formula, then* $\Gamma \cup \{A\}$ *or* $\Gamma \cup \{\neg A\}$ *is consistent.*

Proof. First we prove (C1). 'if': Assume that every finite subset of Γ is consistent. Suppose that Γ is not consistent. Then there is $A \in \Gamma$ s.t. $\Gamma \vdash A$ and $\Gamma \vdash \neg A$. So, since every syntactical entailment contains a finite number of inference steps, there are finite $\Gamma' \subset \Gamma$ and $\Gamma'' \subset \Gamma$ s.t. $\Gamma' \vdash A$ and $\Gamma'' \vdash \neg A$. But $\Gamma' \cup \Gamma'' \subset \Gamma$ is finite and entails A as well as $\neg A$, contradicting the finite consistency. We conclude that Γ must be consistent. 'only if': Assume that Γ is consistent. Suppose that there are a finite subset Γ' of Γ and a formula A s.t. $\Gamma' \vdash A$ and $\Gamma' \vdash \neg A$. Then clearly $\Gamma \vdash A$ and $\Gamma \vdash \neg A$, contradicting Γ's consistency. We conclude that every finite subset of Γ must be consistent.

Now we prove (C2). Suppose that Γ is consistent, but $\Gamma \cup \{A\} \vdash \bot$ and $\Gamma \cup \{\neg A\} \vdash \bot$. Then, by Deduction Theorem, $\Gamma \vdash A \to \bot$ and $\Gamma \vdash \neg A \to \bot$. Note that the first entailment is equivalent to $\Gamma \vdash \neg A \vee \bot$. Now, by Disjunction Elimination, we have $\Gamma \vdash \bot \vee \bot$. So $\Gamma \vdash \bot$, contradicting Γ's consistency. We conclude that $\Gamma \cup \{A\}$ or $\Gamma \cup \{\neg A\}$ must be consistent. $\square$

Using the two properties (C1) and (C2), we will prove Lindenbaum's Theorem, which states that there is a maximal consistent set of formulas.

Theorem 6.3 (Lindenbaum)**.** *Let* Γ *be a consistent set of formulas. Then there is a set of formulas* Δ *s.t.*

(a) $\Gamma \subset \Delta$*,*
(b) Δ *is consistent,*
(c) for every formula A*, either* $A \in \Delta$ *or* $\neg A \in \Delta$*.*

Proof. Let X be the set of all formulas and let $\mathcal{A} = \{X' \subset X : X' \text{ is consistent}\}$. By (C1), $\mathcal{A}$ has finite character, and by (C2), $\mathcal{A}$ has the extention property with respect to $\neg$. So we apply the enriched RTT on Γ and obtain $\Delta \in \mathcal{A}$ that satisfies (1), (2) and (3). □

With Lindenbaum's Theorem, we can derive the Model Existence Theorem.

Theorem 6.4 (Model Existence)**.** *Let* Γ *be a consistent set of formulas. Then* Γ *is satisfiable.*

Proof. By Lindenbaum's Theorem, there is a set Δ of formulas such that

(a) $\Gamma \subset \Delta$,
(b) Δ is consistent,
(c) for every formula A, either $A \in \Delta$ or $\neg A \in \Delta$.

Define $\phi : \mathrm{FOR} \to \{T, F\}$ by $\phi(A) = T \Leftrightarrow A \in \Delta$.

We verify that ϕ is a truth assignment. First we check (1). Suppose, for contradiction, that $\phi(A) = \phi(\neg A)$ for some formula A. Then $A, \neg A \in \Delta$, contradicting the consistency of Δ. Now we check (2), i.e. for all formulas A and B, $\phi(A \vee B) = T$ if and only if $\phi(A) = T$ or $\phi(B) = T$. 'only if': Suppose that $\phi(A \vee B) = T$. Then $A \vee B \in \Delta$. Suppose, for contradiction, that $A, B \notin \Delta$. Then by (c), $\neg A, \neg B \in \Delta$. Thus $\neg(A \vee B) = \neg A \wedge \neg B \in \Delta$, contradicting the consistency of Δ. Thus $A \in \Delta$ or $B \in \Delta$, as desired. 'if': Since by the premise $A \in \Delta$ or $B \in \Delta$, $A \vee B \in \Delta$. Therefore $\phi(A \vee B) = T$, as desired.

Now, since ϕ is defined to satisfy Δ, ϕ satisfies $\Gamma \subseteq \Delta$. This proves the Model Existence. □

To prove the Compactness Theorem, we first prove the following lemma, saying that finite satisfiability implies consistency. For the proof, we will assume the validity of the Soundness Theorem.

Lemma 6.5. *Let* Γ *be a finitely satisfiable set of formulas. Then* Γ *is consistent.*

Proof. Suppose, for contradiction, that Γ is not consistent. Since every syntactical entailment contains a finite number of inference steps, there is finite $\Gamma' \subset \Gamma$ s.t. $\Gamma' \vdash \bot$. But then, by Soundness Theorem, $\Gamma' \vDash \bot$, contradicting Γ's finitely satisfiability. Therefore we conclude that Γ is consistent. □

Theorem 6.6 (Compactness)**.** *Let* Γ *be a finitely satisfiable set of formulas. Then* Γ *is satisfiable.*

Proof. By Lemma 6.5, Γ is consistent. So, by the Model Existence Theorem, Γ is satisfiable. □

7 References

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