Strict bilimits

with an overview of limit notions in 2-categories

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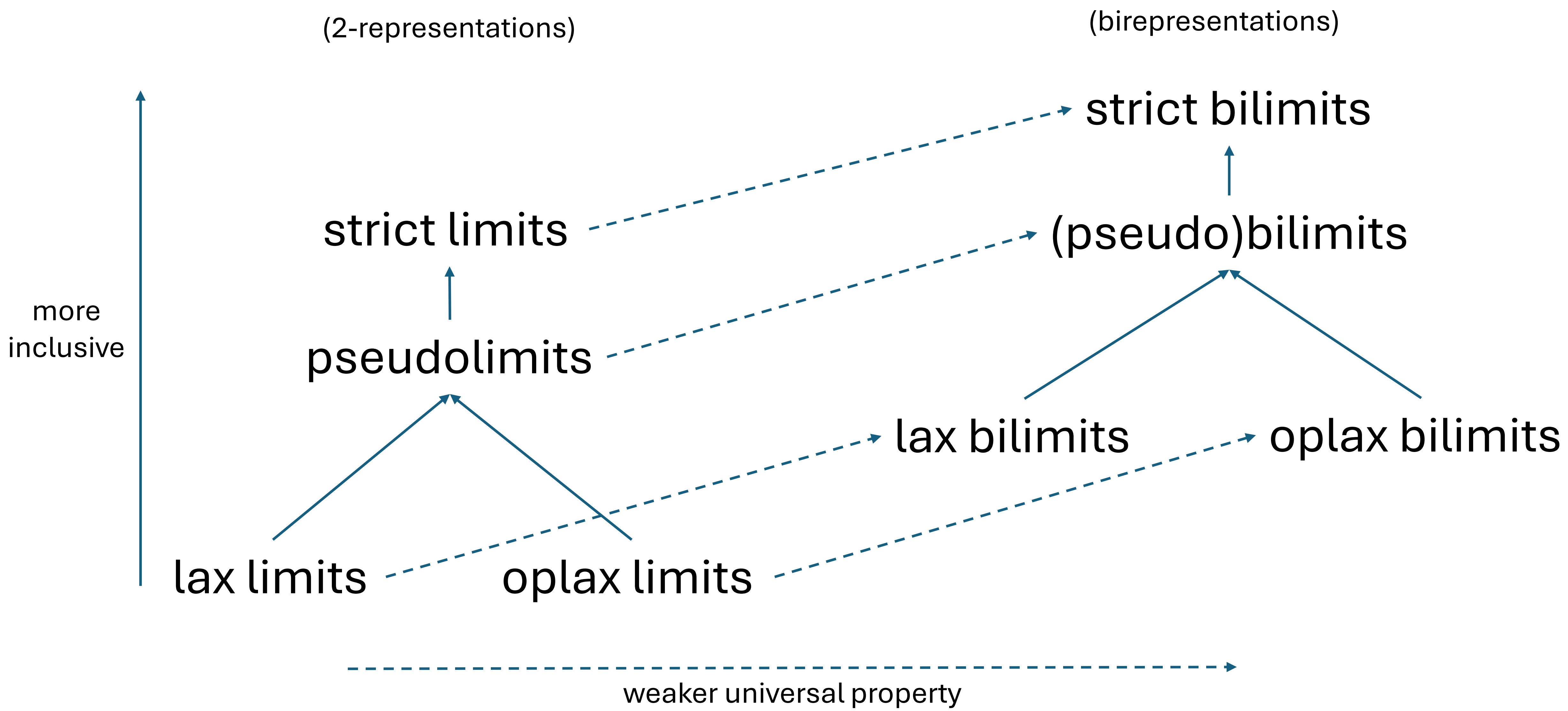
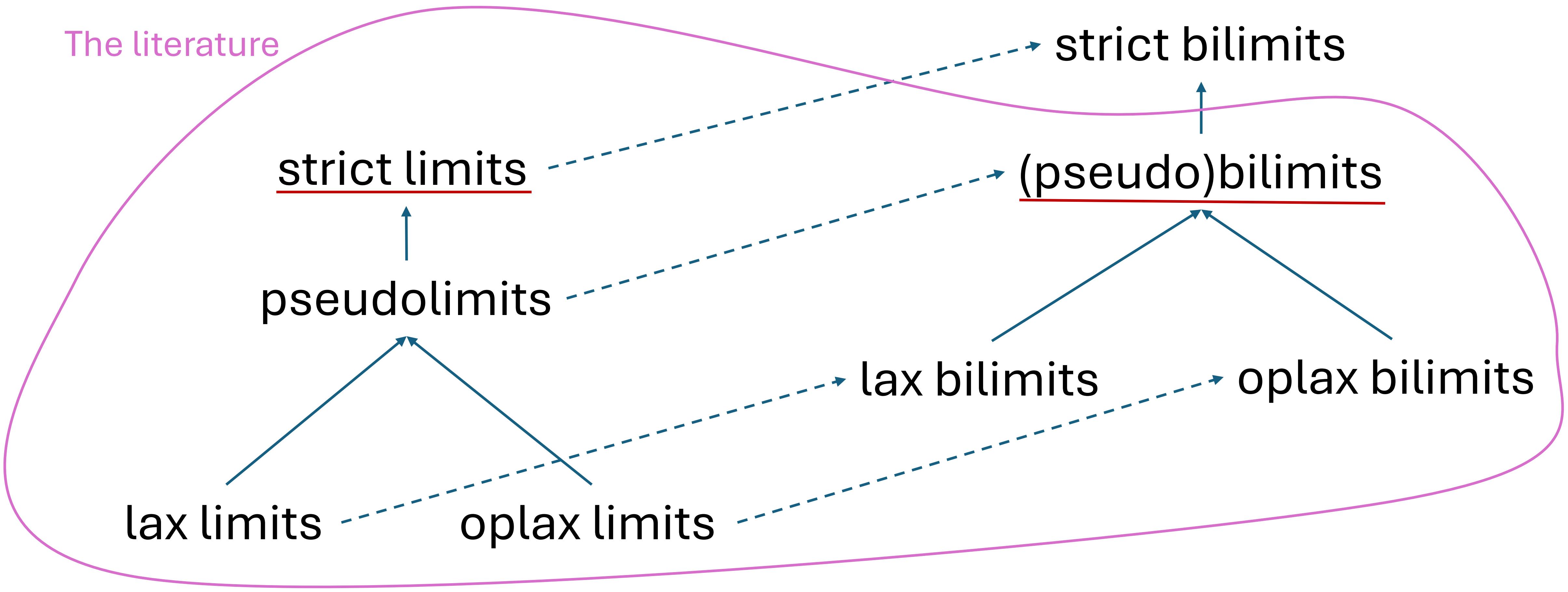
rather than bicategories!!

Overview: variants of weighted limits in 2-categories



The literature is rather silent about strict bilimits, while they are the most general.

Question: are they "unnecessary", or do they have proper examples?



Plan

1. xxx limits vs xxx bilimits
2. Definitions of strict, pseudo, lax and oplax (bi)limits
    - Examples
3. Strict subsumes pseudo, lax and oplax
4. A class of strict bilimits ‘admitting’ another
5. Pseudobilimits don’t admit biequalisers (which is a strict bilimit)
6. There is a biequaliser that cannot be given as an equaliser

1. xxx limits vs xxx bilimits (1)

Definition. Let K be a category. A *representation* of a functor $F: K \to Set$ consists of an object $r \in K$ together with an isomorphism

$$\rho: K(r, -) \cong F$$

in the functor category $[K, Set]$.

Example. Let A and K be categories. A *limit* of a functor (*diagram*) $d: A \to K$ is a representation of the functor

$$K^{op} \to Set: k \mapsto [A, K](\Delta_k, d).$$

If $W: A \to Set$ is a functor (*weight*), a W-*weighted limit* of d is a representation of the functor

$$K^{op} \to Set: k \mapsto [A, Set](W, K(k, d-)).$$

1. xxx limits vs xxx bilimits (2)

Definition. Let K be a 2-category. A *2-representation* of a 2-functor $F\colon K \to Cat$ shall refer to a Cat-enriched representation of F, that is, an object $r \in K$ together with an isomorphism

$$\rho\colon K(r,-) \xrightarrow{\cong} F \tag{1}$$

in $[K, Cat]_{\mathrm{s,s}}$.

Definition. Let K be a 2-category. A *birepresentation* of a 2-functor $F\colon K \to Cat$ is an object $r \in K$ together with an equivalence

$$\rho\colon K(r,-) \xrightarrow{\simeq} F \tag{2}$$

in $[K, Cat]_{\mathrm{s,p}}$.

Beware: **two** changes from a 2-representation!

2. Definitions of strict/pseudo/lax/oplax (bi)limit

Let A and K be 2-categories, and let $W: A \to Cat$ and $d: A \to K$ be 2-functors.

Definition (in words)**.** Let foo = strict, pseudo, lax or oplax.

- A W-weighted foo *limit* of d is a 2-representation for the *Cat*-valued contravariant 2-functor on K of W-weighted foo cones on d.
- A W-weighted foo *bilimit* of d is a birepresentation for the *Cat*-valued contravariant 2-functor on K of W-weighted foo cones on d.

More precisely (strict bilimit):

Definition. A W-weighted *strict bilimit* of d is a birepresentation of the 2-functor

$$K^{\mathrm{op}} \to Cat\colon k \mapsto [A, Cat]_{\mathrm{s,s}}(W, K(k, d-)).$$

2^+. Examples of 2-dimensional limits

- Conical limits [(conical) strict limits]
- Inserters [non-conical strict limit]
- Equifiers [non-conical strict limit]
- Pseudopullbacks [(conical) pseudolimit]
- Grothendieck construction [(conical) oplax colimit]
  - The Grothendieck construction on a pseudofunctor *F*: *C* → *Cat* is equivalently the oplax colimit of *F*.

$$\mathrm{Lax}(F, \Delta X) \simeq [\textstyle\int F, X]$$

- Indiscrete cats in $MonCat_p$ [foo bicolimit but not foo colimit]
  - $MonCat_p$ has no initial object: there are always at least two strong monoidal functors into Iso, the walking isomorphism.
  - Easy: 1 is a bi-initial object in $MonCat_p$.
  - Objects equivalent to 1 in $MonCat_p$ are precisely the indiscrete categories.

3. Strict subsumes pseudo, lax and oplax

Will use this as a black box

"Two-dimensional monad theory"

"Flexible limits for 2-categories"

Theorem (special case of Blackwell et al. 1989, Theorem 3.16 for pseudo and lax; Bird et al. 1989, p. 7 for oplax). If A is a small 2-category, then the three inclusion 2-functors

$$[A, Cat]_{\mathrm{s,s}} \hookrightarrow [A, Cat]_{\mathrm{s,p}}, [A, Cat]_{\mathrm{s,l}}, [A, Cat]_{\mathrm{s,o}}$$

have left adjoints $Q_{\mathrm{p}}, Q_{\mathrm{l}}, Q_{\mathrm{o}}$ respectively.

What follows: deduce from this that strict bilimits subsume pseudo, lax and oplax bilimits.

When K is a 2-category, let $[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}$ denote the wide and locally full sub-2-category of $[K^{\mathrm{op}}, Cat]_{\mathrm{s,p}}$ on equivalences.

Corollary. Let A be a small 2-category, K a locally small 2-category, and $W\colon A \to Cat$ and $d\colon A \to K$ 2-functors. Let $\mathrm{foo} \in \{\mathrm{p(seudo)}, \mathrm{l(ax)}, \mathrm{o(plax)}\}$. For each 0-cell $r \in K$, there is an isomorphism of categories[9]

$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}\left(K(-,r), \boxed{[A, Cat]_{\mathrm{s,s}}\Big(Q_{\mathrm{foo}}(W), \lambda a.\, K(-,da)\Big)}\right)$$
$$\cong$$
$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}\left(K(-,r), \boxed{[A, Cat]_{\mathrm{s,foo}}\Big(W, \lambda a.\, K(-,da)\Big)}\right).$$

That is, in simplified words, a W-weighted foo bilimit of d with vertex r is precisely a $Q_{\mathrm{foo}}(W)$-weighted strict bilimit of d with vertex r. This way, strict bilimits subsume pseudo, lax and oplax bilimits.

Corollary (abridged)**.** There is an isomorphism of categories

$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}(K(-,r), [A, Cat]_{\mathrm{s,s}}\Big(Q_{\mathrm{foo}}(W), \lambda a.\, K(-,da)\Big))$$

$$\cong$$

$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}(K(-,r), [A, Cat]_{\mathrm{s,foo}}\Big(W, \lambda a.\, K(-,da)\Big)).$$

Remark. We can substitute 'p' with 's' and 'eqv' with 'iso' above, and obtain that that strict limits subsume pseudo, lax and oplax limits.

Remark. Pseudo(bi)limits subsume lax and oplax (bi)limits, by an analogous mechanism (details in the post).

4. A class of strict bilimits ‘admitting’ another

Let $\mathcal{V}, \mathcal{W}$ be classes of **weights**, that is, pairs (A, W) where A is a 2-category and $W: A \to Cat$ is a 2-functor.

Inclusion between such classes is **not** a desirable way to capture the idea that one class of strict bilimits ‘covers’ another, since a larger class of strict bilimits may be constructed from a smaller class of strict bilimits.

Definition. We say $\mathcal{V}$ (weakly) **admits $\mathcal{W}$ as classes of strict limits** if every 2-category that has strict limits of type $\mathcal{V}$ admits strict limits of type $\mathcal{W}$.

We say $\mathcal{V}$ (weakly) **admits $\mathcal{W}$ as classes of strict bilimits** if every 2-category that has strict bilimits of type $\mathcal{V}$ admits strict bilimits of type $\mathcal{W}$.

Example (Bird et al. 1989, Proposition 2.1). Products, inserters and equifiers admit (as strict limits) all pseudo, lax and oplax limits.

5. Pseudobilimits don't admit biequalisers

Let $MonCat_{\mathrm{p}}$ denote the 2-category of monoidal categories and strong monoidal functors.

Proposition. $MonCat_{\mathrm{p}}$ does not have strict biequalisers.

Proof. Consider the diagram $\{0\} \underset{1}{\overset{0}{\rightrightarrows}} \{0,1\}$ in $MonCat_{\mathrm{p}}$, where $\{0\}$ and $\{0,1\}$ are regarded as indiscrete monoidal categories (with any choice of a monoidal structure on $\{0,1\}$). Clearly no monoidal category can be the vertex of a cone on this diagram, because every monoidal category is inhabited. In particular, this diagram has no strict bilimit. This proves the proposition. ∎

Since we know $MonCat_{\mathrm{p}}$ is a pseudobilimit-complete 2-category (it is in fact pseudolimit-complete; see Blackwell et al. 1989, Theorem 2.6), it is an example of a pseudobilimit-complete 2-category that does not have strict biequalisers. (In particular, it is an example of a pseudobilimit-complete 2-category that is not strict-limit complete.) Therefore:

Corollary. Pseudobilimits don't weakly admit strict biequalisers. In particular, they don't weakly admit strict bilimits. ∎

6. There is a biequaliser that cannot be given as an equaliser

Now, given a 2-category K, a 2-category K' will be constructed that (for suitable choices of K) has no equalisers but has biequalisers.

Construction. Let K be a 2-category. We will define a 2-category K'.

The 0-cells of K' are the 0-cells of K. For each 1-cell $a\colon x \to y$ in K, its two copies $a^0, a^1\colon x \to y$ are 1-cells in K', and all 1-cells in K' are of this form. The 2-cells $f^p \to g^q$ $(p, q \in \{0, 1\})$ in K' are the 2-cells $f \to g$ in K.

The identity 1-cell on a 0-cell $x \in K'$ is the 1-cell id_x^0. If $f^p\colon x \to y$ and $g^q\colon y \to z$ are 1-cells, then their composite is $g^q f^p := (gf)^{\max\{p,q\}}\colon x \to z$. The identity as well as vertical and horizontal composite 2-cells in K' are given by the respective operations in K. This defines K'.

Properties of K'

1. K' is a 2-category.

Proof. 1. The composition of 1-cells is associative, for $\max\{-_1, -_2\}$ is associative. Identity 1-cells are unital, for 0 is unital with respect to $\max\{-_1, -_2\}$. The vertical and horizontal compositions of 2-cells are associative, and identity 2-cells are unital, because the same is the case for the underlying 2-cells in K. For the likewise reason, the horizontal composition of 2-cells preserves identity 2-cells as well as vertical composition. Therefore K' is a 2-category.

2. The forgetful 2-functor $u\colon K' \to K$ is a biequivalence of 2-categories.

Proof. The 2-functor $u\colon K' \to K$ is bijective on 0-cells, 1-homwise surjective and 2-homwise bijective, hence a biequivalence .

From: Johnson and Yau (2021)

Theorem 7.4.1 (Whitehead Theorem for Bicategories). *A pseudofunctor of bicategories $F : \mathsf{B} \longrightarrow \mathsf{C}$ is a biequivalence if and only if F is*

1. *(1) essentially surjective on objects,*
2. *(2) essentially full on 1-cells, and*
3. *(3) fully faithful on 2-cells.*

3. Let $W\colon A \to Cat$ be a 2-functor. If K has strict W-(co)limits, then K' has strict W-bi(co)limits.

(Below essentially proves that a biequivalence lifts bilimits)

Proof. Let $d'\colon A \to K'$ be a 2-functor.

Let $l \in K_0$ and an equivalence of categories (there is in fact an iso of categories)

$$K(x,l) \simeq [A, Cat]_{\mathrm{s,s}}(W, K(x, ud'-))$$

cones on *ud'* in *K* with vertex *x*

strictly natural in $x \in K_0$ be a strict W-limit of $ud'\colon A \to K$. Let l' be the unique 0-cell in K' such that $l = ul'$. Then we have the chain of equivalences of categories

$$\begin{aligned} K'(x',l') \simeq K(ux',ul') = K(ux',l) &\simeq [A, Cat]_{\mathrm{s,s}}(W, K(ux', ud'-)) \\ &\simeq [A, Cat]_{\mathrm{s,s}}(W, K'(x', d'-)) \end{aligned} \qquad (\star)$$

strictly natural in $x' \in K'_0$, providing the 0-cell $l' \in K'$ with the structure of a strict W-bilimit of d'. This proves 3.

3. Let $W\colon A \to Cat$ be a 2-functor. If K has strict W-(co)limits, then K' has strict W-bi(co)limits.

Proof of (★). In light of 2., we have an equivalence of categories, i.e. an equivalence in the 2-category Cat,

$$K'(x', y') \simeq K(ux', uy')$$

that is strictly natural in $x', y' \in K'_0$. It follows that we have an equivalence

$$K'(x', d'-) \simeq K(ux', ud'-)$$

in the 2-category $[A, Cat]_{\mathrm{s,s}}$ that is strictly natural in $x' \in K'_0$. This induces an equivalence of categories

cones on *d'* in *K'* with vertex *x*

$$[A, Cat]_{\mathrm{s,s}}(W, K'(x', d'-)) \simeq [A, Cat]_{\mathrm{s,s}}(W, K(ux', ud'-))$$

cones on *ud'* in *K* with vertex *ux'*

that is strictly natural in $x' \in K'_0$.

4. Diagrams of the form $x \overset{f^0}{\underset{g^1}{\rightrightarrows}} y$ admits no strict equaliser in K'.

Proof. Let $c \overset{h^p}{\rightarrow} x$ be a strict cone on the diagram, then necessarily $p = 1$.

Can such a strict cone ever be a limit? Now whenever $i^q \colon c \to c$ is a 1-cell such that the triangle in

$$\begin{array}{ccccc} c & & & & \\ {\scriptstyle i^q}\downarrow & \searrow^{h^1} & & & \\ c & \xrightarrow[h^1]{} & x & \overset{f^0}{\underset{g^1}{\rightrightarrows}} & y \end{array}$$

commutes, then the triangle in

$$\begin{array}{ccccc} c & & & & \\ {\scriptstyle i^{1-q}}\downarrow & \searrow^{h^1} & & & \\ c & \xrightarrow[h^1]{} & x & \overset{f^0}{\underset{g^1}{\rightrightarrows}} & y \end{array}$$

must also commute. Therefore no strict cone on the diagram can satisfy the uniqueness condition of 2-universality. This proves 4.

Corollary.

1. If K is inhabited, then K' does not have strict equalisers.
2. If K is inhabited and has strict equalisers, then K' has strict biequalisers but lacks strict equalisers.
3. If K is strict-limit complete, then K' is strict-bilimit complete but lacks strict equalisers (so is not strict-limit complete).

Proof.

1. As soon as a 0-cell $x \in K'$ exists, the diagram

$$x \underset{\mathrm{id}_x^1}{\overset{\mathrm{id}_x^0}{\rightrightarrows}} x$$

can be formed, which admits no strict equaliser by Property 4.

Corollary.

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Proof.

2. Immediate by 1. and Property 3.
3. Since K is strict-limit complete, it has a limit of the empty diagram, so is inhabited. Hence also immediate by 1. and Property 3. This proves the corollary.

Corollary.

1. If K is inhabited, then K' does not have strict equalisers.
2. If K is inhabited and has strict equalisers, then K' has strict biequalisers but lacks strict equalisers.
3. If K is strict-limit complete, then K' is strict-bilimit complete but lacks strict equalisers (so is not strict-limit complete).

Therefore K' gives the desired 2-category having biequalisers but no equalisers, as long as K is inhabited and has equalisers. ■

For concrete examples of K', we can take:

- *K* := 1, which is inhabited and evidently has all strict limits, in particular equalisers.
- *K* := *Cat*, which is inhabited and known also to have all strict limits.

Question. Is there a "naturally occurring" example of a strict bilimit that is not weakly admissible by pseudobilimits and not equivalent to a strict limit?

- John Bourke told me at CT2024 that Bourke, Lack and Vokřínek (2023), "Adjoint functor theorems for homotopically enriched categories" considers '*E*-weak coequalisers' for *E* the class of surjective equivalences in *Cat*: they are coequalisers whose universal property is given in terms of surjective equivalences of categories, hence should be proper examples of strict bi(co)limits.

Thank you!

All details and references are available in the post
“Strict bilimit and its proper examples”
on my website (sorilee.github.io)