Strict bilimits

with an overview of limit notions in 2-categories

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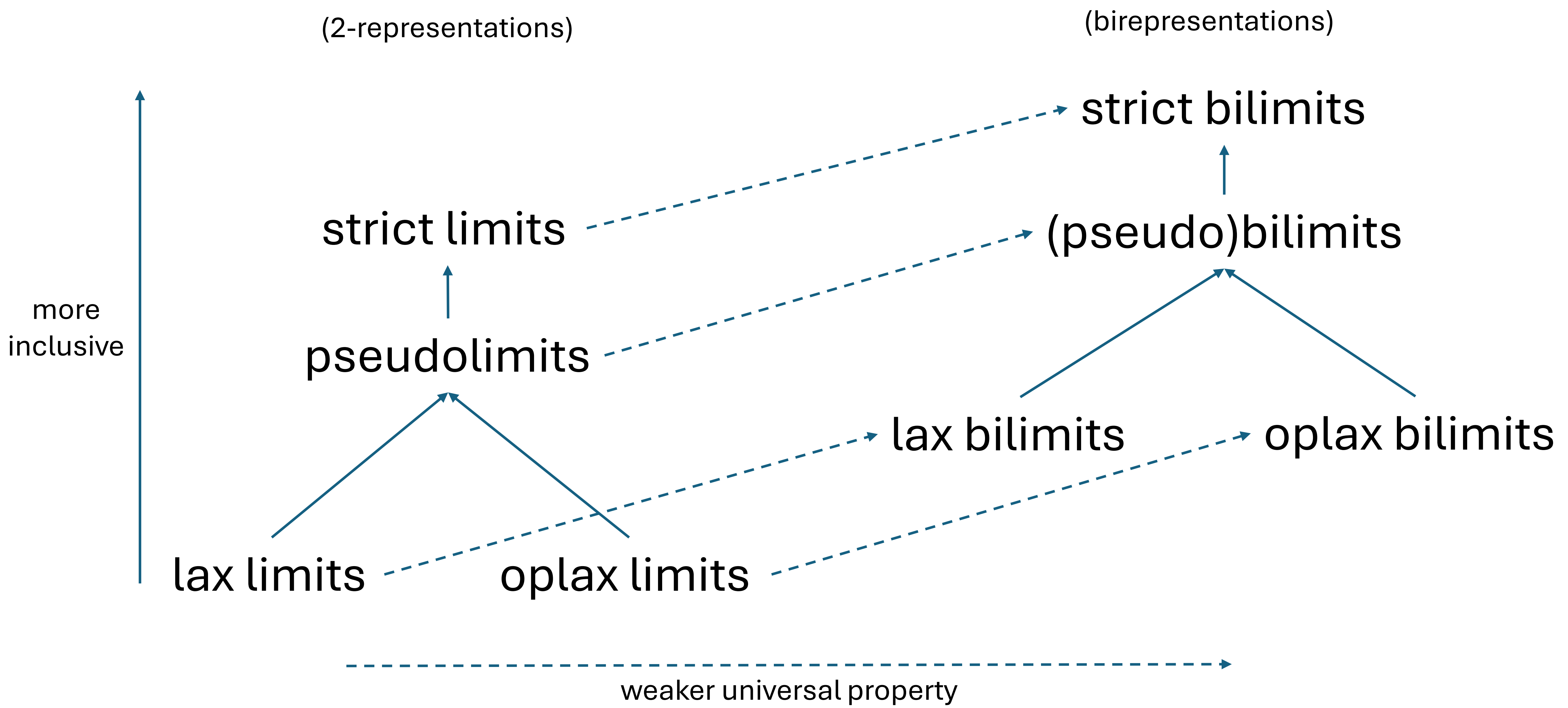
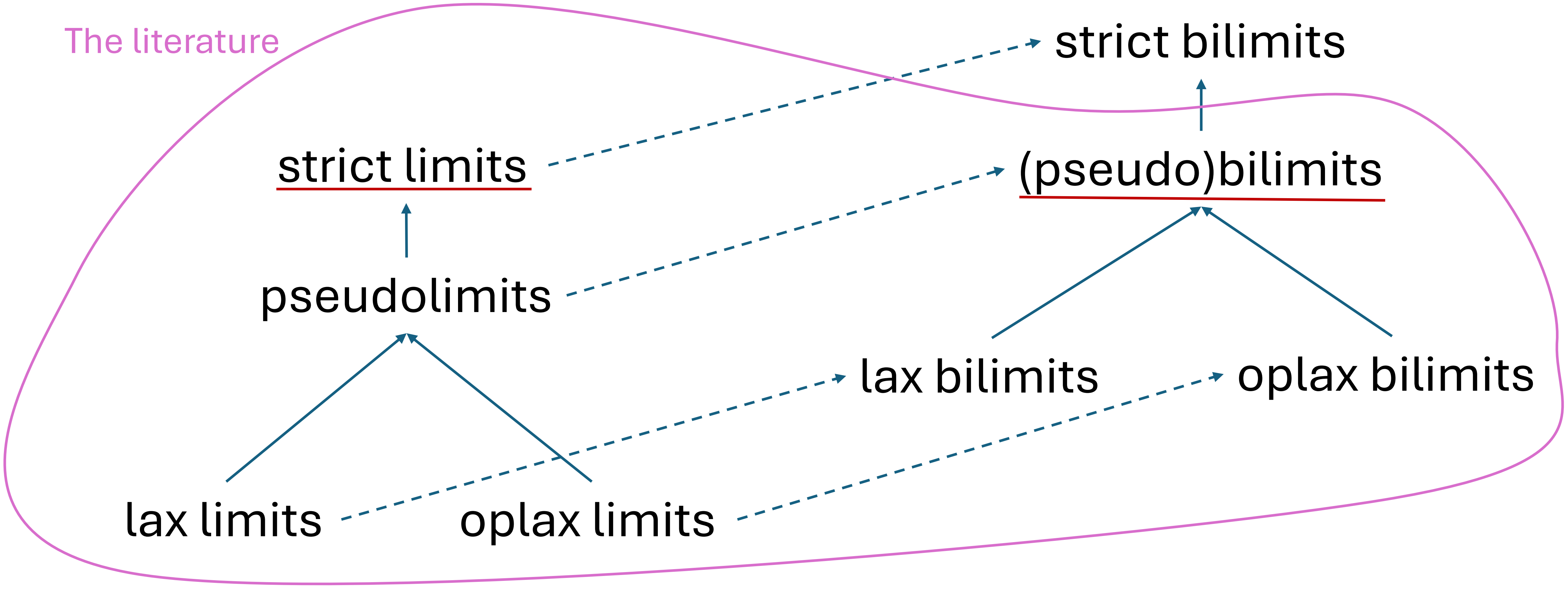
Overview: variants of weighted limits in 2-categories

rather than bicategories!!



The literature is rather silent about strict bilimits, while they are the most general.

Question: are they "unnecessary", or do they have proper examples?



Main observation

Main Proposition. There are 2-categories A and K', and 2-functors $W: A \to Cat$ and $d: A \to K'$, such that

1. d has a W-weighted strict bilimit,
2. d has no W-weighted strict limit, and ← "not covered by strict limits"
3. the weight W is not weakly admitted (see below) by bilimits. ← "not covered by bilimits"

Part 1: Preliminaries

1. 2-categories (via enriched categories)
2. 2-functors (via enriched functors)
3. Strict, pseudo- and (op)lax natural transformations
4. Bicategories and some examples (← was asked about in a previous seminar)

1. Enriched categories – definition

Let V be a monoidal category.

A $\mathcal{V}$*-category* $\mathcal{A}$ consists of a set ob $\mathcal{A}$ of *objects*, a *hom-object* $\mathcal{A}(A,B) \in \mathcal{V}_0$ for each pair of objects of $\mathcal{A}$, a *composition law* $M = M_{ABC} \colon \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,C)$ for each triple of objects, and an *identity element* $j_A \colon I \longrightarrow \mathcal{A}(A,A)$ for each object; subject to the associativity and unit axioms expressed by the commutativity of

$$\begin{array}{ccc}
(\mathcal{A}(C,D) \otimes \mathcal{A}(B,C)) \otimes \mathcal{A}(A,B) & \xrightarrow{\quad a \quad} & \mathcal{A}(C,D) \otimes (\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)) \\
\Big\downarrow {\scriptstyle M \otimes 1} & & \Big\downarrow {\scriptstyle 1 \otimes M} \\
\mathcal{A}(B,D) \otimes \mathcal{A}(A,B) & & \mathcal{A}(C,D) \otimes \mathcal{A}(A,C) \\
\searrow {\scriptstyle M} & & \swarrow {\scriptstyle M} \\
& \mathcal{A}(A,D), &
\end{array} \tag{1.3}$$

$$\begin{array}{ccccc}
\mathcal{A}(B,B) \otimes \mathcal{A}(A,B) & \xrightarrow{M} & \mathcal{A}(A,B) & \xleftarrow{M} & \mathcal{A}(A,B) \otimes \mathcal{A}(A,A) \\
\Big\uparrow {\scriptstyle j \otimes 1} & \nearrow {\scriptstyle l} & & \nwarrow {\scriptstyle r} & \Big\uparrow {\scriptstyle 1 \otimes j} \\
I \otimes \mathcal{A}(A,B) & & & & \mathcal{A}(A,B) \otimes I.
\end{array} \tag{1.4}$$

From: Kelly (2005)

1. Enriched categories – examples

- *V* = *Cat* => *V*-categories = 2-categories
- *V* = *Set* => *V*-categories = categories
- *V* = *2Cat* => *V*-categories = 3-categories
- *V* = $(k\text{-}Mod, \otimes)$ => *V*-categories = *k*-linear categories
  - k : a commutative ring
- *V* = *Ch*(*A*) => *V*-categories = dg-categories
  - A : a pre-additive category (:= Ab-enriched category)
- *V* = *sSet* => *V*-categories = 'simplicial categories'
- *V* = *PreOrd* => *V*-categories = preorder-enriched categories
- *V* = $(\{0,1\}, \wedge)$ => V-categories = preorders

2. Enriched functors – definition

For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{V}$-*functor* $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ consists of a function

From: Kelly (2005)

$$T\colon \operatorname{ob}\mathcal{A} \longrightarrow \operatorname{ob}\mathcal{B}$$

together with, for each pair $A, B \in \operatorname{ob}\mathcal{A}$, a map $T_{AB}\colon \mathcal{A}(A,B) \longrightarrow \mathcal{B}(TA, TB)$, subject to the compatibility with composition and with the identities expressed by the commutativity of

$$\begin{CD}
\mathcal{A}(B,C) \otimes \mathcal{A}(A,B) @>{M}>> \mathcal{A}(A,C) \\
@V{T \otimes T}VV @VV{T}V \\
\mathcal{B}(TB,TC) \otimes \mathcal{B}(TA,TB) @>>{M}> \mathcal{B}(TA,TC),
\end{CD} \tag{1.5}$$

$$\begin{array}{ccc}
 & & \mathcal{A}(A,A) \\
 & \overset{j}{\nearrow} & \\
I & & \Big\downarrow T \\
 & \underset{j}{\searrow} & \\
 & & \mathcal{B}(TA,TA).
\end{array} \tag{1.6}$$

Definition. A *2-functor* is a *Cat*-functor.

3. Enriched natural transformations – definition

For $\mathcal{V}$-functors $T, S \colon \mathcal{A} \longrightarrow \mathcal{B}$, a *$\mathcal{V}$-natural transformation* $\alpha \colon T \longrightarrow S \colon \mathcal{A} \longrightarrow \mathcal{B}$ is an ob $\mathcal{A}$-indexed family of *components* $\alpha_A \colon I \longrightarrow \mathcal{B}(TA, SA)$ satisfying the $\mathcal{V}$-naturality condition expressed by the commutativity of

$$\begin{array}{ccccc} & I \otimes \mathcal{A}(A,B) & \xrightarrow{\alpha_B \otimes T} & \mathcal{B}(TB,SB) \otimes \mathcal{B}(TA,TB) & \\ {\scriptstyle l^{-1}} \nearrow & & & & \searrow {\scriptstyle M} \\ \mathcal{A}(A,B) & & & & \mathcal{B}(TA,SB). \\ {\scriptstyle r^{-1}} \searrow & & & & \nearrow {\scriptstyle M} \\ & \mathcal{A}(A,B) \otimes I & \xrightarrow[S \otimes \alpha_A]{} & \mathcal{B}(SA,SB) \otimes \mathcal{B}(TA,SA) & \end{array} \qquad (1.7)$$

From: Kelly (2005)

Definition. A *2-natural transformation* or *strict natural transformation* is a *Cat*-natural transformation.

3. (Op)lax, pseudo and strict natural transformations

Given (possibly weak) 2-categories C, D and (possibly lax or oplax) 2-functors $F, G: C \to D$, a ***lax natural transformation*** $\alpha: F \Rightarrow G$ is given by

From: nLab

- for each $A \in C$ a 1-morphism $\alpha_A: F(A) \to G(A)$ in D, as usual
- for each $f: A \to B$ in C a 2-morphism $\alpha_f: G(f) \circ \alpha_A \Rightarrow \alpha_B \circ F(f)$:

$$
\begin{array}{ccc}
FA & \xrightarrow{F(f)} & FB \\
{\scriptstyle \alpha_A}\downarrow & \Rightarrow & \downarrow{\scriptstyle \alpha_B} \\
GA & \xrightarrow[G(f)]{} & GB
\end{array}
$$

+ compatibility axioms: 'functoriality' of α_A in A and 'naturality' of α_f in f.

For **oplax natural**, reverse the direction of α_f.
For **pseudo natural**, require α_f to be an isomorphism.
For **strict natural**, require α_f to be identity.

4. Bicategories (1) – Definition

From: Johnson and Yau (2021)

A *bicategory* is a tuple $(\mathsf{B}, 1, c, a, \ell, r)$

Objects: B is equipped with a class $\mathsf{Ob}(\mathsf{B}) = \mathsf{B}_0$, whose elements are called *objects* or 0-*cells* in B. If $X \in \mathsf{B}_0$, we also write $X \in \mathsf{B}$.

Hom Categories: For each pair of objects $X, Y \in \mathsf{B}$, B is equipped with a category $\mathsf{B}(X, Y)$, called a *hom category*.

Identity 1-Cells: For each object $X \in \mathsf{B}$, a functor $1_X : \mathbf{1} \longrightarrow \mathsf{B}(X, X)$

Horizontal Composition: For each triple of objects $X, Y, Z \in \mathsf{B}$, a functor

$$c_{XYZ} : \mathsf{B}(Y, Z) \times \mathsf{B}(X, Y) \longrightarrow \mathsf{B}(X, Z)$$

Associator: For objects $W, X, Y, Z \in \mathsf{B}$,

$$a_{WXYZ} : c_{WXZ}(c_{XYZ} \times \mathrm{Id}_{\mathsf{B}(W,X)}) \longrightarrow c_{WYZ}(\mathrm{Id}_{\mathsf{B}(Y,Z)} \times c_{WXY})$$

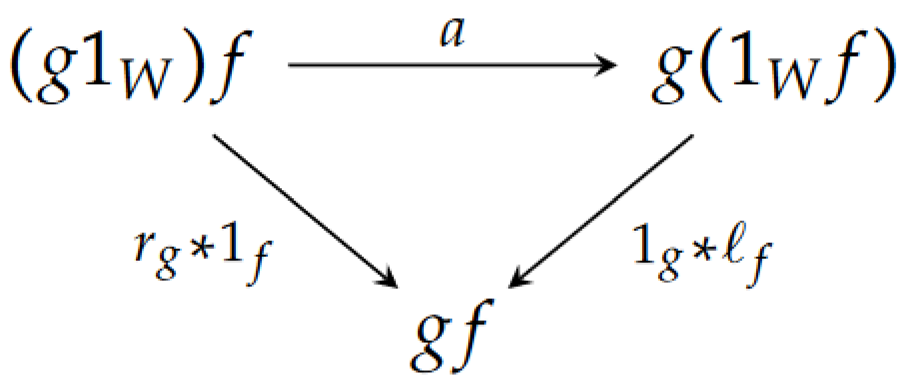
Unitors: For each pair of objects $X, Y \in \mathsf{B}$,

$$c_{XYY}(1_Y \times \mathrm{Id}_{\mathsf{B}(X,Y)}) \xrightarrow{\ell_{XY}} \mathrm{Id}_{\mathsf{B}(X,Y)} \xleftarrow{r_{XY}} c_{XXY}(\mathrm{Id}_{\mathsf{B}(X,Y)} \times 1_X)$$

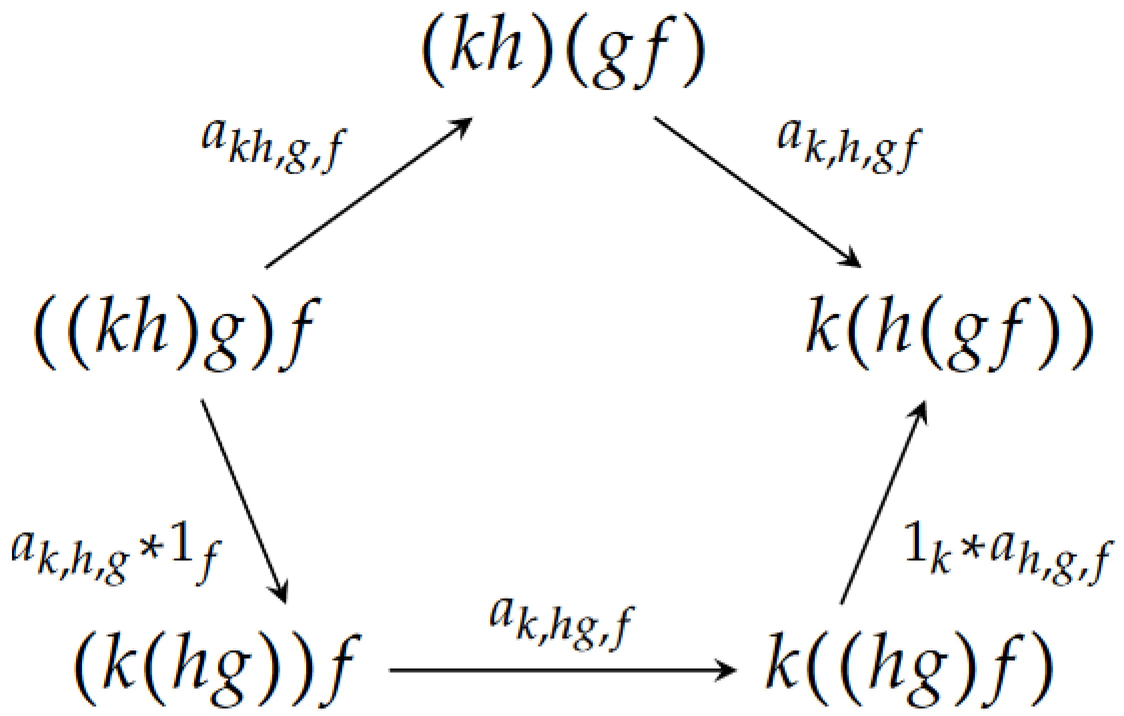
4. Bicategories (2) – Definition (cont'd)

From: Johnson and Yau (2021)

The Unity Axiom:



The Pentagon Axiom:



6. Bicategories (3) – Proper examples

- BiMod, the bicategory of rings and bimodules
  - Objects: Rings
  - Arrows $R \to S$: R-S bimodules
  - 2-cells: bimodule homomorphisms
  - Composition: if *M* is a *R*-*S* bimodule and *N* is a *S*-*T* bimodule, then

$$N \circ M \ := M \otimes_S N$$

Note: The tensor product is not strictly associative, whence a bicategory.

- $\Pi_2(X)$, the fundamental bigroupoid of a topological space X
  - Objects: points in X
  - Arrows: paths
  - 2-cells: homotopies between paths

Note: The usual 'halving' composition of paths is not strictly associative.

Part 2: Strict bilimit and its proper examples

1. 2-representations vs birepresentations
2. Definitions of strict, pseudo, lax and oplax (bi)limits
   - Formalism of weighted limits
   - Examples
3. Strict (bi)limits subsume pseudo, lax and oplax (bi)limits
4. A class of strict bilimits 'admitting' another
5. Strict bilimits don't admit biequalisers
6. There is a biequaliser that cannot be given as an equaliser.

1. 2-representations vs birepresentations (1)

Definition. Let C be a category. A *representation* of a functor $F: C \to Set$ consists of an object $r \in C$ together with an isomorphism

$$\rho: C(r, -) \cong F$$

in the functor category $[C, Set]$.

Example. Let A and K be categories. A *limit* of a functor (*diagram*) $d: A \to K$ is a representation of the functor

$$K^{op} \to Set: x \mapsto [A, K](\Delta_x, d)$$

1. 2-representations vs birepresentations (2)

Definition. Let K be a 2-category. A *2-representation* of a 2-functor $F\colon K \to Cat$ shall refer to a Cat-enriched representation of F, that is, an object $r \in K$ together with an isomorphism

$$\rho\colon K(r, -) \xrightarrow{\cong} F \tag{1}$$

in $[K, Cat]_{s,s}$.

Definition. Let K be a 2-category. A *birepresentation* of a 2-functor $F\colon K \to Cat$ is an object $r \in K$ together with an equivalence

$$\rho\colon K(r, -) \xrightarrow{\simeq} F \tag{2}$$

in $[K, Cat]_{s,p}$.

Beware: **two** changes from a 2-representation!

2. Definitions of strict/pseudo/lax/oplax (bi)limit

Let A and K be 2-categories, and let $W: A \to Cat$ and $d: A \to K$ be 2-functors.

Definition (in words)**.** Let foo = strict, pseudo, lax or oplax.

- A W-weighted foo *limit* of d is a 2-representation for the *Cat*-valued contravariant 2-functor on K of W-weighted foo cones on d.
- A W-weighted foo *bilimit* of d is a birepresentation for the *Cat*-valued contravariant 2-functor on K of W-weighted foo cones on d.

More precisely (strict bilimit):

Definition. A W-weighted *strict bilimit* of d is a birepresentation of the 2-functor

$$K^{\mathrm{op}} \to Cat \colon k \mapsto [A, Cat]_{\mathrm{s,s}}(W, K(k, d-)).$$

2^{α}. Explaining the formalism of a weighted cone

A,K : 2-categories   W: A -> Cat   d: A -> K $\gamma\colon W \Rightarrow K(-, d-)$

A : 'diagram shape'; a 2-category

∋ cells that index the constituents of a diagram

d : 2-functor that projects the diagram shape into the target 2-category

$W(\bullet)$: 'leg shape' (at •); a category

$W(\rightarrow)$: a functor; maps constituents from one leg shape to those from another

$W(\Downarrow)$: a natural transformation – for each object in the domain leg shape, an arrow in the codomain leg shape

$\gamma_\bullet$: 'leg' at •; a functor that projects the leg shape into the target 2-category

N.B. objects in the leg shape ↦ 1-cells, arrows ↦ 2-cells.

2^{β}. Examples of 2-dimensional limits

- Conical limits [(conical) strict limits] [slide]
- Inserters [non-conical strict limit] **[new slide]**
- Equifiers [non-conical strict limit] **[new slide]**
- Pseudopullbacks [(conical) pseudolimit] [BB]
- Grothendieck construction [(conical) oplax colimit]
  - The Grothendieck construction on a pseudofunctor *F: C → Cat* is equivalently the oplax colimit of *F*.

$$\mathrm{Lax}(F, \Delta X) \simeq [\textstyle\int F, X]$$

- Indiscrete cats in $MonCat_p$ [foo bicolimit but not foo colimit]
  - $MonCat_p$ has no initial object: there are always at least two strong monoidal functors into Iso, the walking isomorphism.
  - Easy: 1 is a bi-initial object in $MonCat_p$.
  - Objects equivalent to 1 in $MonCat_p$ are precisely the indiscrete categories.

Conical limits. For $W = \Delta_1$, we have

$$[A, Cat]_{s,s}(W, K(x, d-)) = [A, Cat]_{s,s}(\Delta\_1, K(x, d-)) \cong [A, K]_{s,s}(\Delta_x, d)$$

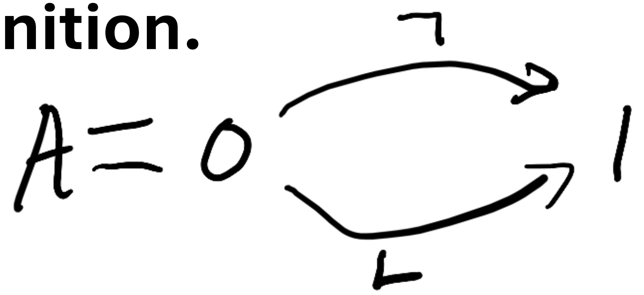
Thus a Δ_1-weighted strict limit of *d* is precisely a ‘conical limit’ of *d*.

Inserters

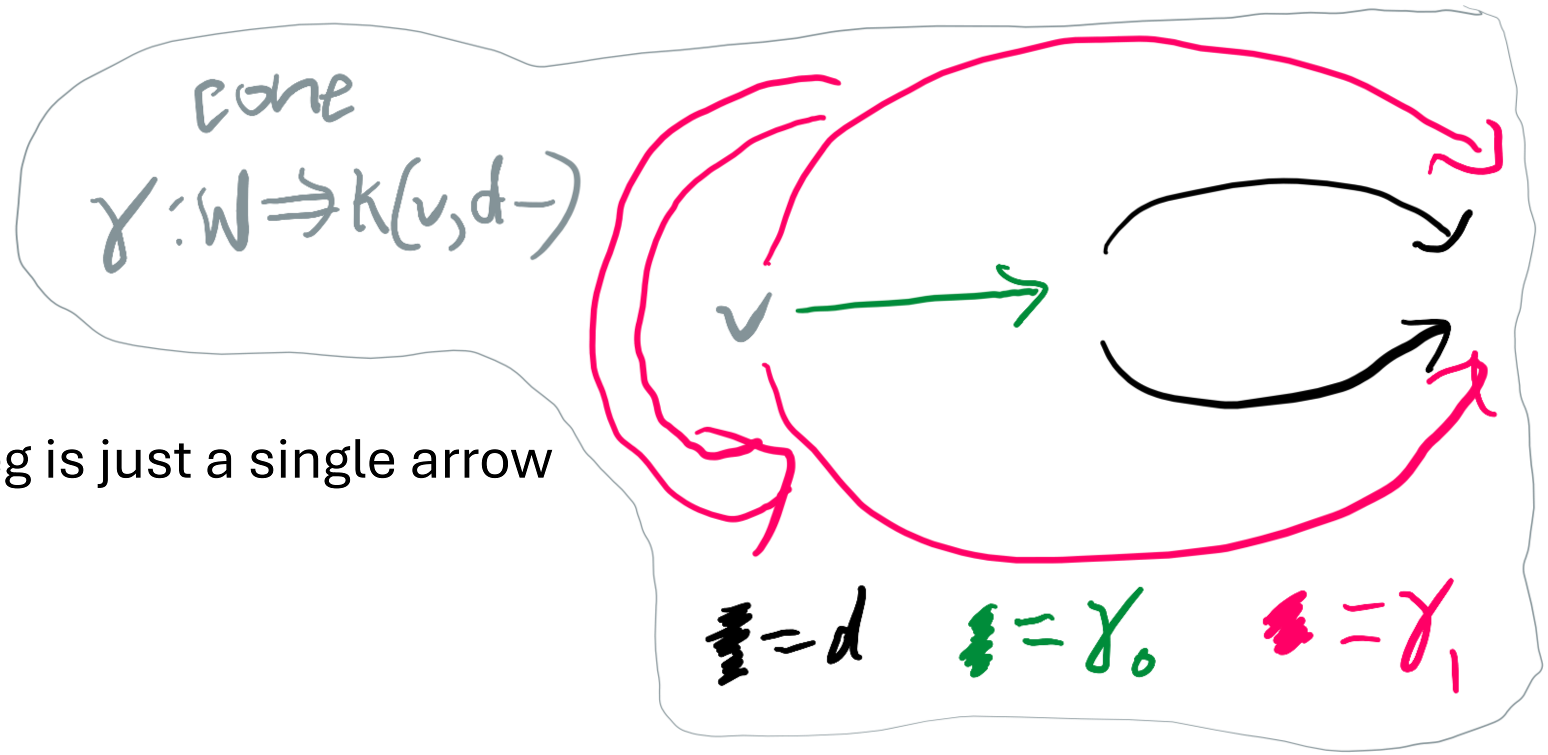
Idea. An **inserter** is a 2-universal 1-cell the precomposition with which "inserts" a 2-cell between a pair of parallel 2-cells.

- So an inserter cone is a lax version of an equaliser cone.

Definition.

- 

- $W(0) = 1$ because this leg is just a single arrow
- $W(1) = \{\cdot \to \cdot\}$
- $W(\text{rest}) = \text{obvious}$



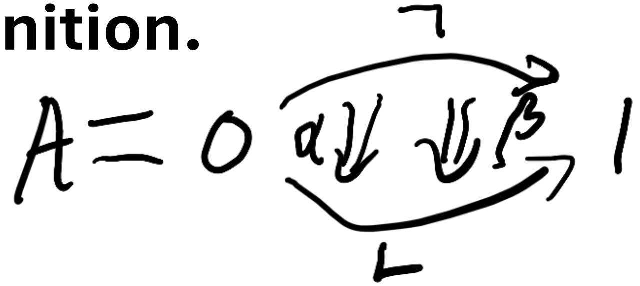
Example in $\boldsymbol{Cat}$. Let $F, G: C \to D$ be functors. The inserter of F and G is given by the category whose objects are pairs (c, b) where $c \in C_0$ and $b: F(c) \to G(c)$, and whose arrows $(c, b) \to (c', b')$ are arrows $a: c \to c'$ in C such that $G(a) \circ b = b' \circ F(a)$.

Equifiers

Idea. An **equifier** is a 2-universal 1-cell the precomposition with which identifies a pair of parallel 2-cells.

- Just like an equaliser (in our 2-dimensional context) is a 2-universal 1-cell the precomposition with which identifies a pair of parallel 1-cells.

Definition.

- 

- $W(0) = 1$ because this leg is just a single arrow
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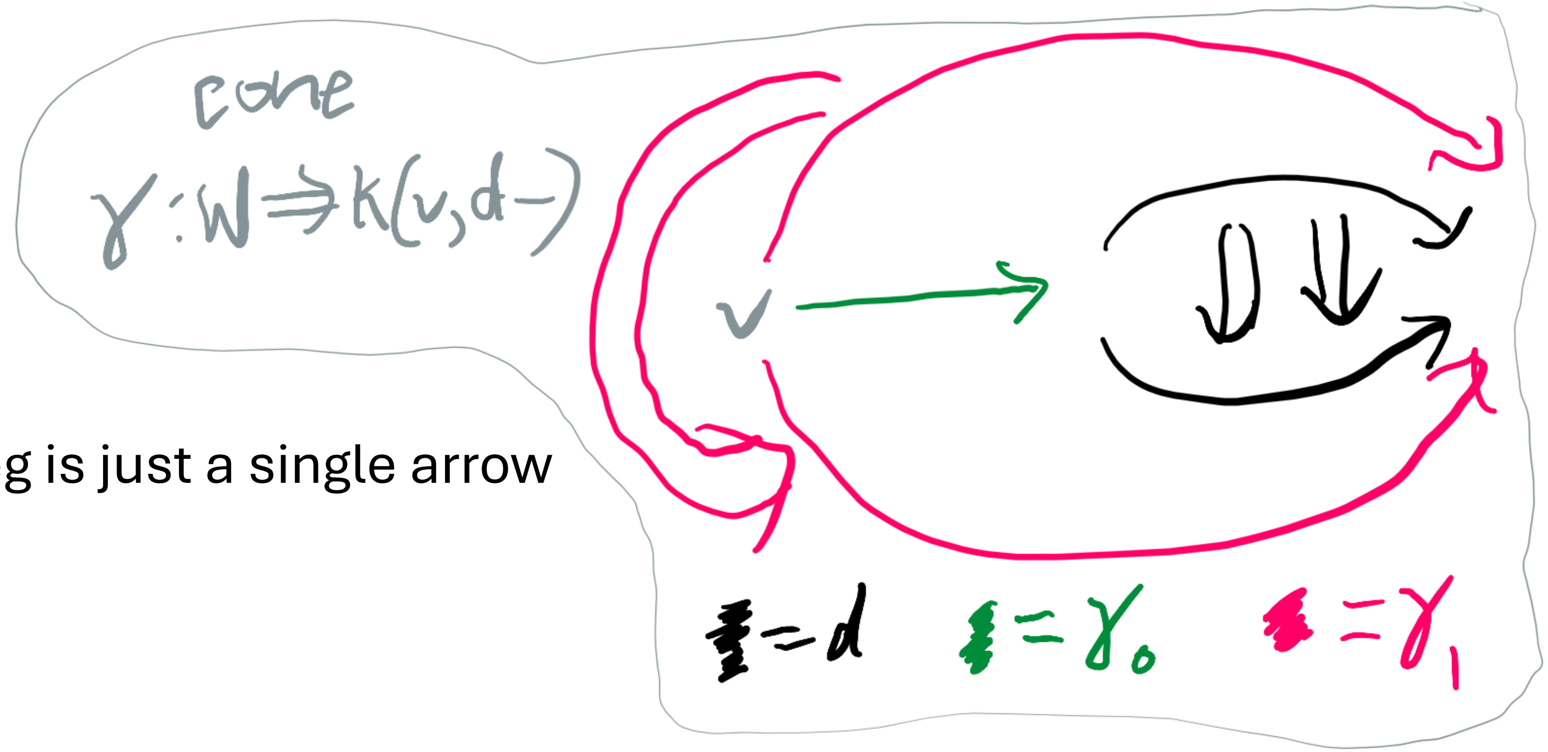


Example in $\mathcal{Cat}$. Let $\theta, \zeta : F \Rightarrow G : C \to D$ be natural transformations. The equifier of θ and ζ is the full subcategory of C consisting of those objects c for which $\theta_c = \zeta_c$.

3. Strict (bi)limits subsume pseudo, lax and oplax (bi)limits

Will use this as a black box

“Two-dimensional monad theory”

“Flexible limits for 2-categories”

Theorem (special case of Blackwell et al. 1989, Theorem 3.16 for pseudo and lax; Bird et al. 1989, p. 7 for oplax)**.** If A is a small 2-category, then the three inclusion 2-functors

$$[A, Cat]_{\mathrm{s,s}} \hookrightarrow [A, Cat]_{\mathrm{s,p}}, [A, Cat]_{\mathrm{s,l}}, [A, Cat]_{\mathrm{s,o}}$$

have left adjoints $Q_{\mathrm{p}}, Q_{\mathrm{l}}, Q_{\mathrm{o}}$ respectively.

What follows: deduce from this that strict bilimits subsume pseudo, lax and oplax bilimits.

When K is a 2-category, let $[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}$ denote the wide and locally full sub-2-category of $[K^{\mathrm{op}}, Cat]_{\mathrm{s,p}}$ on equivalences.

Corollary. Let A be a small 2-category, K a locally small 2-category, and $W\colon A \to Cat$ and $d\colon A \to K$ 2-functors. Let $\mathrm{foo} \in \{\mathrm{p(seudo)}, \mathrm{l(ax)}, \mathrm{o(plax)}\}$. For each 0-cell $r \in K$, there is an isomorphism of categories[9]

$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}\Big(K(-,r), \boxed{[A, Cat]_{\mathrm{s,s}}\Big(Q_{\mathrm{foo}}(W), \lambda a.\, K(-, da)\Big)}\Big)$$

$$\cong$$

$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}\Big(K(-,r), \boxed{[A, Cat]_{\mathrm{s,foo}}\Big(W, \lambda a.\, K(-, da)\Big)}\Big).$$

That is, in simplified words, a W-weighted foo bilimit of d with vertex r is precisely a $Q_{\mathrm{foo}}(W)$-weighted strict bilimit of d with vertex r. This way, strict bilimits subsume pseudo, lax and oplax bilimits.

Corollary (abridged). There is an isomorphism of categories

$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}(K(-,r), [A, Cat]_{\mathrm{s,s}}\Big(Q_{\mathrm{foo}}(W), \lambda a.\, K(-,da)\Big))$$

$$\cong$$

$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}(K(-,r), [A, Cat]_{\mathrm{s,foo}}\Big(W, \lambda a.\, K(-,da)\Big)).$$

Remark. We can substitute ‘p’ with ‘s’ and ‘eqv’ with ‘iso’ above, and obtain that that strict limits subsume pseudo, lax and oplax limits.

Remark. Pseudo(bi)limits subsume lax and oplax (bi)limits, by an analogous mechanism (details in the post).

4. A class of strict bilimits 'admitting' another

Let $\mathcal{V}, \mathcal{W}$ be classes of **weights**, that is, pairs (A, W) where A is a 2-category and $W: A \to Cat$ is a 2-functor.

Inclusion between such classes is **not** a desirable way to capture the idea that one class of strict bilimits 'covers' another, since a larger class of strict bilimits may be constructed from a smaller class of strict bilimits.

Definition. We say $\mathcal{V}$ (weakly) **admits $\mathcal{W}$ as classes of strict limits** if every 2-category that has strict limits of type $\mathcal{V}$ admits strict limits of type $\mathcal{W}$.

We say $\mathcal{V}$ (weakly) **admits $\mathcal{W}$ as classes of strict bilimits** if every 2-category that has strict bilimits of type $\mathcal{V}$ admits strict bilimits of type $\mathcal{W}$.

Example (Bird et al. 1989, Proposition 2.1)**.** Products, inserters and equifiers admit (as strict limits) all pseudo, lax and oplax limits.

5. Pseudobilimits don't admit strict bilimits

Let $MonCat_{\mathrm{p}}$ denote the 2-category of monoidal categories and strong monoidal functors.

Proposition. $MonCat_{\mathrm{p}}$ does not have strict biequalisers.

Proof. Consider the diagram $\{0\} \underset{1}{\overset{0}{\rightrightarrows}} \{0,1\}$ in $MonCat_{\mathrm{p}}$, where $\{0\}$ and $\{0,1\}$ are regarded as indiscrete monoidal categories (with any choice of a monoidal structure on $\{0,1\}$). Clearly no monoidal category can be the vertex of a cone on this diagram, because every monoidal category is inhabited. In particular, this diagram has no strict bilimit. This proves the proposition. ∎

Proposition. $MonCat_{\mathrm{p}}$ does not have strict biequalisers. ∎

Since we know $MonCat_{\mathrm{p}}$ is a pseudobilimit-complete 2-category (it is in fact pseudolimit-complete; see Blackwell et al. 1989, Theorem 2.6), it is an example of a pseudobilimit-complete 2-category that does not have strict biequalisers. (In particular, it is an example of a pseudobilimit-complete 2-category that is not strict-limit complete.) Therefore:

Corollary. Pseudobilimits don't weakly admit strict biequalisers. In particular, they don't weakly admit strict bilimits. ∎

6. There is a biequaliser that cannot be given as an equaliser.

We will now prove the 'main observation':

Main Proposition. There are 2-categories A and K', and 2-functors $W\colon A \to Cat$ and $d\colon A \to K'$, such that

1. d has a W-weighted strict bilimit,
2. d has no W-weighted strict limit, and ← "not covered by strict limits"
3. the weight W is not weakly admitted (see below) by bilimits. ← "not covered by bilimits"

Namely, a 2-category K' will be constructed from a given 2-category K that (for suitable choices of K) has no equalisers but has biequalisers.

Construction. Let K be a 2-category. We will define a 2-category K'.

The 0-cells of K' are the 0-cells of K. For each 1-cell $a\colon x \to y$ in K, its two copies $a^0, a^1\colon x \to y$ are 1-cells in K', and all 1-cells in K' are of this form. The 2-cells $f^p \to g^q$ $(p, q \in \{0, 1\})$ in K' are the 2-cells $f \to g$ in K.

The identity 1-cell on a 0-cell $x \in K'$ is the 1-cell id_x^0. If $f^p\colon x \to y$ and $g^q\colon y \to z$ are 1-cells, then their composite is $g^q f^p := (gf)^{\max\{p,q\}}\colon x \to z$. The identity as well as vertical and horizontal composite 2-cells in K' are given by the respective operations in K. This defines K'.[21]

Proposition.

1. K' is a 2-category.

2. The forgetful 2-functor $u\colon K' \to K$ is a biequivalence of 2-categories.

3. Let $W\colon A \to Cat$ be a 2-functor. If K has strict W-(co)limits, then K' has strict W-bi(co)limits.

4. Diagrams of the form $x \overset{f^0}{\underset{g^1}{\rightrightarrows}} y$ admits no strict equaliser in K'.

1. K' is a 2-category.

Proof. 1. The composition of 1-cells is associative, for $\max\{-_1, -_2\}$ is associative. Identity 1-cells are unital, for 0 is unital with respect to $\max\{-_1, -_2\}$. The vertical and horizontal compositions of 2-cells are associative, and identity 2-cells are unital, because the same is the case for the underlying 2-cells in K. For the likewise reason, the horizontal composition of 2-cells preserves identity 2-cells as well as vertical composition. Therefore K' is a 2-category.

2. The forgetful 2-functor $u\colon K' \to K$ is a biequivalence of 2-categories.

Proof. The 2-functor $u\colon K' \to K$ is bijective on 0-cells, 1-homwise surjective and 2-homwise bijective, hence a biequivalence .

From: Johnson and Yau (2021)

Theorem 7.4.1 (Whitehead Theorem for Bicategories). *A pseudofunctor of bicategories $F : \mathsf{B} \longrightarrow \mathsf{C}$ is a biequivalence if and only if F is*

(1) essentially surjective on objects,
(2) essentially full on 1-cells, and
(3) fully faithful on 2-cells.

3. Let $W \colon A \to Cat$ be a 2-functor. If K has strict W-(co)limits, then K' has strict W-bi(co)limits.

(Below essentially proves that a biequivalence lifts bilimits)

Proof. Let $d' \colon A \to K'$ be a 2-functor.
Let $l \in K_0$ and an equivalence of categories (there is in fact an isomorphism of categories)

$$K(x, l) \simeq [A, Cat]_{\mathrm{s,s}}(W, K(x, ud'-))$$

cones on *ud'* in *K* with vertex *x*

strictly natural in $x \in K_0$ be a strict W-limit of $ud' \colon A \to K$. Let l' be the unique 0-cell in K' such that $l = ul'$. Then we have the chain of equivalences of categories

$$\begin{aligned} K'(x', l') \simeq K(ux', ul') = K(ux', l) &\simeq [A, Cat]_{\mathrm{s,s}}(W, K(ux', ud'-)) \\ &\simeq [A, Cat]_{\mathrm{s,s}}(W, K'(x', d'-)) \end{aligned} \qquad (\star)$$

strictly natural in $x' \in K'_0$, providing the 0-cell $l' \in K'$ with the structure of a strict W-bilimit of d'. This proves 3.

3. Let $W\colon A \to Cat$ be a 2-functor. If K has strict W-(co)limits, then K' has strict W-bi(co)limits.

Proof of (★). In light of 2., we have an equivalence of categories, i.e. an equivalence in the 2-category Cat,

$$K'(x', y') \simeq K(ux', uy')$$

that is strictly natural in $x', y' \in K'_0$. It follows that we have an equivalence

$$K'(x', d'-) \simeq K(ux', ud'-)$$

in the 2-category $[A, Cat]_{\mathrm{s,s}}$ that is strictly natural in $x' \in K'_0$. This induces an equivalence of categories

cones on *d'* in *K'* with vertex *x*

$$[A, Cat]_{\mathrm{s,s}}(W, K'(x', d'-)) \simeq [A, Cat]_{\mathrm{s,s}}(W, K(ux', ud'-))$$

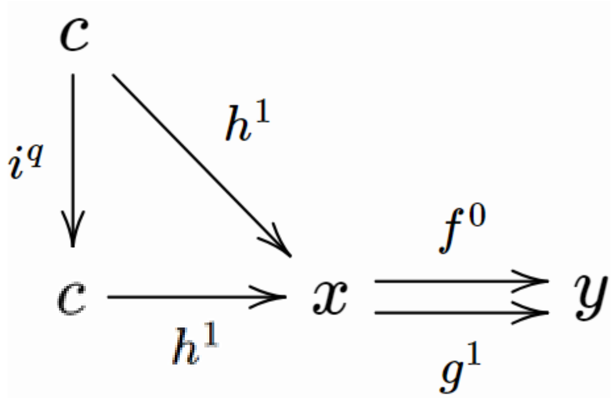
cones on *ud'* in *K* with vertex *ux'*

that is strictly natural in $x' \in K'_0$.

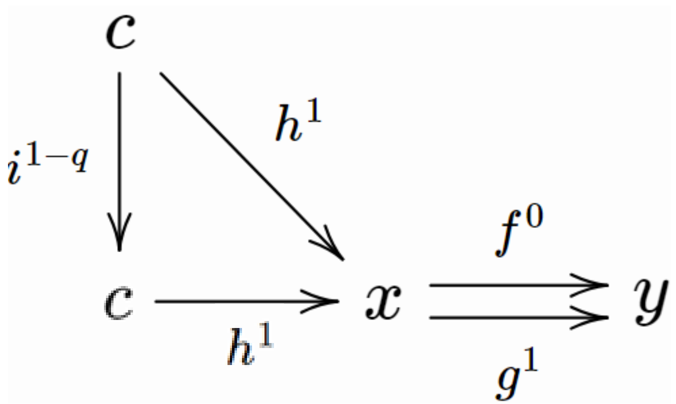
4. Diagrams of the form $x \underset{g^1}{\overset{f^0}{\rightrightarrows}} y$ admits no strict equaliser in K'.

(Succinctly: no cone of this diagram is ‘monic’.)

Proof. Let $c \xrightarrow{h^p} x$ be a strict cone on the diagram, then necessarily $p = 1$. Now whenever $i^q\colon c \to c$ is a 1-cell such that the triangle in



commutes, then the triangle in



must also commute. Therefore no strict cone on the diagram can satisfy the uniqueness condition of 2-universality. This proves 4.

Corollary.

1. If K is inhabited, then K' does not have strict equalisers.
2. If K is inhabited and has strict equalisers, then K' has strict biequalisers but lacks strict equalisers.
3. If K is strict-limit complete, then K' is strict-bilimit complete but lacks strict equalisers (so is not strict-limit complete).

Proof.

1. As soon as a 0-cell $x \in K'$ exists, the diagram

$$x \underset{\mathrm{id}_x^1}{\overset{\mathrm{id}_x^0}{\rightrightarrows}} x$$

can be formed, which admits no strict equaliser by the proposition's 4.

Corollary.

1. If K is inhabited, then K' does not have strict equalisers.
2. If K is inhabited and has strict equalisers, then K' has strict biequalisers but lacks strict equalisers.
3. If K is strict-limit complete, then K' is strict-bilimit complete but lacks strict equalisers (so is not strict-limit complete).

Proof.

2. Immediate by 1. and the proposition's 3.
3. Since K is strict-limit complete, it has a limit of the empty diagram, so is inhabited. Hence also immediate by 1. and the proposition's 3. This proves the corollary.

Main Proposition. There are 2-categories A and K' and 2-functors $W\colon A \to Cat$ and $d\colon A \to K'$ such that

1. d has a W-weighted strict bilimit,
2. d has no W-weighted strict limit, and ← "not covered by strict limits"
3. the weight W is not weakly admitted (see below) by bilimits. ← "not covered by bilimits"

Proof. By item 2. of the Corollary, if *K* is any inhabited 2-category having strict equalisers, then *K'* gives an example: we have seen that *K'* has a parallel pair of arrows [= diagram d]

- that has a strict biequaliser [fulfilling 1.],
- but has no strict equaliser [fulfilling 2.];
- moreover, we know strict biequalisers are not weakly admitted by bilimits [fulfilling 3.].

For concrete examples, we can take:

- *K* := 1, which is inhabited and evidently has all strict limits, in particular strict equalisers.
- *K* := *Cat*, which is inhabited and known also to have all strict limits. ■

Question. Is there a "naturally occurring" example of a strict bilimit that is not weakly admissible by pseudobilimits and not equivalent to a strict limit?

- John Bourke told me at CT2024 that Bourke, Lack and Vokřínek (2023), "Adjoint functor theorems for homotopically enriched categories" considers '*E*-weak coequalisers' for *E* the class of surjective equivalences in *Cat*: they are coequalisers whose universal property is given in terms of surjective equivalences of categories, hence should be proper examples of strict bi(co)limits.

Thank you!

The underlying materials and references are available in the post "Strict bilimit and its proper examples" on sorilee.github.io