Identity types in predicate logic

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1. A (informal) question arising by looking at the history of interpretations of Intensional TT (Martin-Löf, 1970s):
   - Hofmann and Streicher (1994): types as 1-groupoids
   - Homotopy type theory (mid-late 00s): types as ∞-groupoids

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2. A structural question from the categorical perspective:

   "Can identity types be added universally to a model of type theory without them?"

$\rightsquigarrow$ These two thoughts lead to the question:

Does some model construction based on ∞-groupoids add identity types universally to a model of type theory without them?

Overview

In this talk, the last question is considered in a rather simplistic, truncated version of dependent type theory: many-sorted predicate logic, or more precisely indexed preorders.

Does some model construction based on 0-groupoids add "identity types" universally to an indexed preorder without them?

Overview

In this talk, the last question is considered in a rather simplistic, truncated version of dependent type theory: many-sorted predicate logic, or more precisely indexed preorders.

Does some model construction based on 0-groupoids add "identity types" universally to an indexed preorder without them?

Summary of findings:

1. The ER-descent construction adds identity objects universally.
2. The PER-descent construction adds partial identity objects universally.
3. Virtualisation promotes partial identity objects to identity objects, universally (but in a sense different from the previous ones).

Identity objects (1)

Let $P = (P^0, P^1)$ be an indexed $(\wedge, \top)$-preorder over a $\times$-category:

- P^0 is a category with binary products,
- P^1 is a functor $(P^0)^{\mathrm{op}} \to \mathrm{Pre}^{\wedge,\top}$.

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Definition (in internal logic)

An identity object on an $X \in \mathrm{Ob}(P^0)$ is an $\mathrm{Id}_X \in P^1(X \times X)$, s.t.

1. (*introduction*) $x : X \vdash \mathrm{Id}_X(x, x)$, and
2. (*elimination*) for any $Y \in \mathrm{Ob}(P^0)$ and $p, q \in P^1(X \times X \times Y)$, if

$$x : X, y : Y; p(x, x, y) \vdash q(x, x, y),$$

then $x : X, x' : X, y : Y; \mathrm{Id}_X(x, x') \wedge p(x, x', y) \vdash q(x, x', y)$.

We say $P := (P^0, P^1)$ has identity objects if each X has one.

Identity objects (2)

Definition (properly)

An identity object on an $X \in \mathrm{Ob}(P^0)$ is an $\mathrm{Id}_X \in P^1(X \times X)$, s.t.

1. (*introduction*) $\top \leqslant (X \xrightarrow{\delta} X \times X)^*(\mathrm{Id}_X)$, and
2. (*elimination*) for any $Y \in \mathrm{Ob}(P^0)$ and $p, q \in P^1(X \times X \times Y)$, if

$$(X \times Y \xrightarrow{\delta \times Y} X \times X \times Y)^*(p) \leqslant (X \times Y \xrightarrow{\delta \times Y} X \times X \times Y)^*(q),$$

then $(X \times X \times Y \xrightarrow{\pi_1, \pi_2} X \times X)^*(\mathrm{Id}_X) \wedge p \leqslant q$.

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Theorem

An indexed $(\wedge, \top)$-poset over a finite-product category has identity objects if and only if it has 'Lawvere equality', i.e. is an 'elementary doctrine'.

The ER-descent construction

Theorem (Pasquali 2015)

There is a 2-functor $\mathrm{ER}\colon \mathrm{IdxPos}_{\mathrm{sn}}^{\times,1,\wedge,\top} \to \mathrm{ED}$ *that is right 2-adjoint to inclusion.*

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Construction (Maietti-Rosolini-Pasquali)

The indexed preorder $\mathrm{ER}(P)$ is given by:

- Objects in $\mathrm{ER}(P)^0$: equivalence relations $(X, \sim)$ in P
- Arrows $(X, \sim_X) \to (Y, \sim_Y)$: $X \xrightarrow{f} Y$ s.t. $\sim_X \leqslant (f \times f)^*(\sim_Y)$
- $\mathrm{ER}(P)^1(X, \sim) \coloneqq \{p \mid \pi_1^*(p) \wedge \sim \leqslant \pi_2^*(p)\} \overset{\text{full}}{\subset} P^1(X)$.

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Scholium

The assignment $P \mapsto \mathrm{ER}(P)$ *extends to a 2-functor* $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top} \to \mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top,\mathrm{Id}}$ *that is right biadjoint to the inclusion 2-functor.*

PER-descent construction

Let P be an indexed $\wedge$-preorder over a binary-product category.

Definition

Define $\mathrm{PER}(P)$ in the same way as $\mathrm{ER}(P)$, but with as objects in $\mathrm{PER}(P)^0$ *partial* equivalence relations $(X, \sim)$ in P instead.

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Example (Tripos-to-topos construction)

$$\text{Triposes} \xrightarrow[\mathrm{PER}]{(1)} \xrightarrow[\text{'virtualise'}]{(2)} \xrightarrow[\text{take functional relations as arrows}]{(3)} \xrightarrow[\text{identify arrows by extensionality}]{(4)} \xrightarrow[\text{take underlying category}]{(5)} \text{Toposes}$$

Partial identity objects (1)

Definition (in internal logic)

We say P has partial identity objects if each object $X \in P^0$ is *equipped* with an element $\mathrm{PId}_X \in P^1(X \times X)$, such that

1. (*partial reflexivity*) $\mathrm{PId}_X(x, x') \vdash \mathrm{PId}_X(x, x), \mathrm{PId}_X(x', x')$,
2. (*paravirtual elimination*) for any $Y \in P^0$ and $p, q \in P^1(X \times X \times Y)$, if

   $$\mathrm{PId}_X(x, x) \wedge \mathrm{PId}_Y(y, y) \wedge p(x, x, y) \vdash q(x, x, y),$$

   then $\mathrm{PId}_Y(y, y) \wedge \mathrm{PId}_X(x, x') \wedge p(x, x', y) \vdash q(x, x', y)$,
3. each arrow $f : X \to Y$ satisfies $\mathrm{PId}_X(x, x') \vdash \mathrm{PId}_Y(f(x), f(x'))$,
4. $\mathrm{PId}_{X \times Y}(x, y, x', y') \dashv\vdash \mathrm{PId}_X(x, x') \wedge \mathrm{PId}_Y(y, y')$.

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4. $\mathrm{PId}_{X \times Y}(x, y, x', y') \dashv\vdash \mathrm{PId}_X(x, x') \wedge \mathrm{PId}_Y(y, y')$.

Remark

Choice of identity objects give partial identity objects.

Partial identity objects (2)

Definition (properly)

We say P has partial identity objects if each object $X \in P^0$ is *equipped* with an element $\mathrm{PId}_X \in P^1(X \times X)$, such that

1. (*partial reflexivity*) $\mathrm{PId}_X \leqslant \langle \pi_1, \pi_1 \rangle^*(\mathrm{PId}_X), \langle \pi_2, \pi_2 \rangle^*(\mathrm{PId}_X)$,
2. (*paravirtual elimination*) for any $Y \in P^0$ and $p, q \in P^1(X \times X \times Y)$, if
$$(X \times Y \overset{\pi_1,\pi_1}{\to} X \times X)^*(\mathrm{PId}_X) \wedge (X \times Y \overset{\pi_2,\pi_2}{\to} Y \times Y)^*(\mathrm{PId}_Y) \wedge$$
$$(X \times Y \overset{\delta \times Y}{\to} X \times X \times Y)^*(p) \leqslant (X \times Y \overset{\delta \times Y}{\to} X \times X \times Y)^*(q),$$
then $\langle \pi_3, \pi_3 \rangle^*(\mathrm{PId}_Y) \wedge \langle \pi_1, \pi_2 \rangle^*(\mathrm{PId}_X) \wedge p \leqslant q$,
3. each arrow $f \colon X \to Y$ in P^0 satisfies $\mathrm{PId}_X \leqslant (f \times f)^*(\mathrm{PId}_Y)$,
4. $\mathrm{PId}_{X \times Y} \simeq \langle \pi_1, \pi_3 \rangle^*(\mathrm{PId}_X) \wedge \langle \pi_2, \pi_4 \rangle^*(\mathrm{PId}_Y)$.

PER-descent adds partial identity objects universally

Theorem
The assignment $P \mapsto \mathrm{PER}(P)$ extends to a 2-functor $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge} \to \mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\mathrm{PId}}$ *that is right biadjoint to the inclusion 2-functor.*

Virtualisation (1)

Let P be an *oplaxly sectioned* indexed preorder: each object $X \in P^0$ is equipped with an element $\mathrm{os}_X \in P^1(X)$, and every arrow $f\colon X \to Y$ in P^0 satisfies $\mathrm{os}_X \leqslant f^*(\mathrm{os}_Y)$.

Example

An indexed preorder with partial identity objects is oplaxly sectioned, with $\mathrm{os}_X \coloneqq (X \xrightarrow{\delta} X \times X)^*(\mathrm{PId}_X)$.

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Definition

We define $\mathrm{Virt}(P)$ to be the indexed preorder with $\mathrm{Virt}(P)^0 \coloneqq P^0$ and $\mathrm{Virt}(P)^1(X) \coloneqq (\mathrm{U}_{\mathrm{Set}} P^1(X), \overset{\mathrm{v}}{\leqslant})$ where $p \overset{\mathrm{v}}{\leqslant} q$ if and only if $\mathrm{os}_X \wedge p \leqslant q$.

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Proposition

$\mathrm{Virt}(P)^1$ *is in fact a Kleisli as well as EM object for a (necessarily idempotent) comonad* $\mathrm{v}\colon P^1 \to P^1$ *in the* Pre*-category* $[(P^0)^{\mathrm{op}}, \mathrm{Pre}^\wedge]_{\mathrm{oplax}}$ *given by* $\mathrm{v}_X(p) \coloneqq \mathrm{os}_X \wedge p$.

Virtualisation (2)

Remark

1. The os_X become tops in $\mathrm{Virt}(P)$, as $\mathrm{os}_X \wedge p \leqslant \mathrm{os}_X$ for any p.
2. If P has partial identity objects, then the PId_X become identity objects in $\mathrm{Virt}(P)$, for:
   - Id-introduction for PId_X just means $\delta^*(\mathrm{PId}_X)$ is a top.
   - Id-elimination for PId_X in $\mathrm{Virt}(P)$ is equivalent to paravirtual elimination in P, under the other three axioms of partial identity.

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Theorem

The assignment $P \mapsto \mathrm{Virt}(P)$ extends to

1. *a 2-functor $\mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\mathrm{os}} \to \mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\top}$, as well as*
2. *a 2-functor $\mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\mathrm{PId}} \to \mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\top,\mathrm{Id}}$*

ambidextrously biadjoint to 'the' respective inclusion 2-functor.

The left-biadjoint part also holds with respect to pseudonatural morphisms.

Discussions

1. The ER-descent construction is also a left-biadjoint completion, for adding quotients. (Maietti-Rosolini 2013)

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3. Does a model construction based on 1-groupoids universally add identity types to a model of '1-truncated' type theory?

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More details: for now, the short paper on `sorilee.github.io`