Limits in indexed profunctors over 2-categories

An abstract theory of limits

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Motivation

A spinoff from formal category theory:

- Identify an **axiomatic context** for reasoning about limits.

Example situation

Let C be a locally small category and $c \in \mathrm{Ob}(C)$. We know:

$$C(\lim d, c) \overset{\cong}{\to} \lim C(d, c) \text{ is } \textbf{natural} \text{ in } c,$$

when $d \colon I \to C$ is a diagram; and

$$C(\textstyle\int_I d, c) \overset{\cong}{\to} \textstyle\int_I C(d, c) \text{ is } \textbf{natural} \text{ in } c,$$

when $d \colon I^{\mathrm{op}} \times I \to C$ is an end diagram.

- How do we prove these, illuminating the idea that the two are one phenomenon? We usually do this by a careful **reduction**.
- Another approach: use a **common abstraction**. More useful when there're more limit-like concepts and more propositions.
  - This talk: the structure to prove common properties like above.

Idea

Recall: a **profunctor** $R \nrightarrow S$ is a functor $R^{\text{op}} \times S \to \textsc{Set}$.

Let I, X categories. There is a profunctor $\mathrm{Cone}_X \colon X \nrightarrow \textsc{Cat}(I, X)$,

$$\mathrm{Cone}_X(x, d) := \textsc{Cat}(I, X)(\Delta_x, d).$$

Now, a conical **limit** of d is a pair $(x \in \mathrm{Ob}X, c \in \mathrm{Cone}_X(x, d))$ that is a universal element of the functor $\mathrm{Cone}_X(-, d) \colon X^{\text{op}} \to \textsc{Set}$.

Definition

Let $H \colon R \nrightarrow S$ be a profunctor. A **limit** of $s \in \mathrm{Ob}S$ in H is a pair $(r \in \mathrm{Ob}R, h \in H(r, s))$ that is a universal element of the functor $H(-, s) \colon R^{\text{op}} \to \textsc{Set}$. **A colimit of r is a universal element (s, h) in $H(r, -) \colon S \to \textsc{Set}$.**

Are we done? No, like 0-cells are not the end of formal cat. th.

- For the naturality of limit preservation or the 'adjoints preserve' theorem, need a 2-categorical **'naturality'** of Cone_X in $X \in \textsc{Cat}$.
- Such a 2-natural family of profunctors is the idea of an **indexed profunctor over a 2-category**.

Definition of indexed profunctor

Definition

Let $\mathcal{X}$ be a strict 2-category. An strict covariant **indexed category** R over $\mathcal{X}$ is a strict 2-functor $R_{(-)} \colon \mathcal{X} \to \mathrm{CAT}$.

Definition

Let R, S be indexed categories over $\mathcal{X}$.
An **indexed profunctor** $H \colon R \nrightarrow S$ consists of

1. a profunctor $H_X \colon R_X \nrightarrow S_X$ for each 0-cell $X \in \mathcal{X}$, and
2. a natural transformation of profunctors $H_f \colon H_X \to H_Y(R_f, S_f)$

$$\begin{array}{ccc} R_X & \xrightarrow{H_X} & S_X \\ {\scriptstyle R_f}\downarrow & \Downarrow H_f & \downarrow{\scriptstyle S_f} \\ R_Y & \xrightarrow[H_Y]{} & S_Y \end{array}$$

(whose codomain 'niche' stands for the restriction $H_Y(R_f, S_f) \colon R_X \nrightarrow S_X$)

for each 1-cell $f \colon X \to Y$ in $\mathcal{X}$,

that satisfy coherence axioms (next slide).

Remark: A generalisation of indexed profunctor over a 1-category.

Coherence axioms

1. **Functoriality** of H_X in X:

$$\begin{array}{ccc} R_X & \xrightarrow{H_X} & S_X \\ {\scriptstyle R_{\mathrm{id}_X}}\downarrow & \Downarrow H_{\mathrm{id}_X} & \downarrow{\scriptstyle S_{\mathrm{id}_X}} \\ R_X & \xrightarrow[H_X]{} & S_X \end{array} \quad = \quad \begin{array}{ccc} R_X & \xrightarrow{H_X} & S_X \\ \| & \Downarrow \mathrm{id}_{H_X} & \| \\ R_X & \xrightarrow[H_X]{} & S_X, \end{array}$$

$$\begin{array}{ccc} R_X & \xrightarrow{H_X} & S_X \\ {\scriptstyle R_{g\circ f}}\downarrow & \Downarrow H_{g\circ f} & \downarrow{\scriptstyle S_{g\circ f}} \\ R_Z & \xrightarrow[H_Z]{} & S_Z \end{array} \quad = \quad \begin{array}{ccc} R_X & \xrightarrow{H_X} & S_X \\ {\scriptstyle R_f}\downarrow & \Downarrow H_f & \downarrow{\scriptstyle S_f} \\ R_Y & \xrightarrow[H_Y]{} & S_Y \\ {\scriptstyle R_g}\downarrow & \Downarrow H_g & \downarrow{\scriptstyle S_g} \\ R_Z & \xrightarrow[H_Z]{} & S_Z. \end{array}$$

2. **Naturality** of H_f in f:

$$R_f \left(\begin{array}{cccc} R_X & \xrightarrow{H_X} & S_X & \\ \overset{R_\theta}{\Rightarrow} \ {\scriptstyle }\downarrow{\scriptstyle R_{f'}} & \Downarrow H_{f'} & \downarrow{\scriptstyle S_{f'}} & \\ R_Y & \xrightarrow[H_Y]{} & S_Y & \end{array} \right. \quad = \quad \left. \begin{array}{ccc} R_X & \xrightarrow{H_X} & S_X \\ {\scriptstyle R_f}\downarrow & \Downarrow H_f & {\scriptstyle S_f}\downarrow \ \overset{S_\theta}{\Rightarrow} \\ R_Y & \xrightarrow[H_Y]{} & S_Y \end{array} \right) S_{f'}$$

whenever $\theta \colon f \to f'$ is a 2-cell.

Pointwise: if $r \overset{h}{\rightarrow} s$ in H_X, then

$$\begin{array}{ccc} R_f r & \xrightarrow{H_f h} & S_f s \\ {\scriptstyle R_\theta r}\downarrow & & \downarrow{\scriptstyle S_\theta s} \\ R_{f'} r & \xrightarrow[H_{f'} h]{} & S_{f'} s \end{array}$$

('heteromorphic' diagram!)

'commute' in H_Y.

Examples of indexed profunctor

Example (Conical limits in categories)

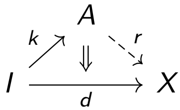
Let $I \in \mathrm{CAT}_0$. Take $\mathcal{X} = \mathrm{CAT}$, $R_X = X$, $S_X = \mathrm{CAT}(I, X)$, $H_X(x, d) = \mathrm{CAT}(I, X)(\Delta_x, d)$. Limits in $H \leftrightarrow$ conical limits.

Example (Ends in categories)

Let $I \in \mathrm{CAT}_0$. Take $\mathcal{X} = \mathrm{CAT}$, $R_X = X$, $S_X = \mathrm{CAT}(I^{\mathrm{op}} \times I, X)$, $H_X(x, d) = \{\text{wedges } x \to d\}$. Limits in $H \leftrightarrow$ ends.

Example (Right Kan extensions in a 2-category $\mathcal{X}$)

Let $I, A \in \mathcal{X}_0$. Take $R_X = \mathcal{X}(A, X)$, $S_X = \mathcal{X}(I, A)^{\mathrm{op}} \times \mathcal{X}(I, X)$, $H_X(r, (k, d)) = \mathcal{X}(I, X)(rk, d)$.



Limits in $H \leftrightarrow$ right Kan extensions.

There are (**probably many**) more of them.

Application: Right 1-cells preserve limits of any type

Let $H\colon R \nrightarrow S$ be an **indexed** profunctor over a 2-category $\mathcal{X}$.

Definition

A 1-cell $f\colon X \to Y$ in $\mathcal{X}$ *preserves limits* of type H if whenever an $h\colon r \Rightarrow s$ in H_X is a limit, then so is $H_f(h)\colon R_f(r) \Rightarrow S_f(s)$ in H_Y.

Remark. Suppose $\lambda_{S_f(s)}\colon \lim S_f(s) \Rightarrow S_f(s)$ is a chosen limit. Then $H_f(h)\colon R_f(r) \Rightarrow S_f(s)$ limit $\Leftrightarrow \overline{H_f(H)}\colon R_f(r) \to \lim S_f(s)$ iso.

As an application, we can state and prove:

Theorem

Any right adjoint 1-cell $f\colon X \to Y$ in $\mathcal{X}$ preserves limits of type H.

- The proof uses every single axiom of an indexed profunctor.

Application: Naturality of limit preservation (1)

To show: $C(\lim d, c) \to \lim C(d, c)$ natural in c, for lim of any type.

True even if the 1-cell in question isn't $C(-, c)$ or limit-preserving:

Theorem

*Let $H\colon R \nrightarrow S$ be an **indexed** profunctor over a 2-category $\mathcal{X}$. Let C be a category, $X, Y \in \mathcal{X}$ objects and $\phi\colon C \to \mathcal{X}(X, Y)$ a functor. Suppose X and Y have chosen limits of type H and let $s \in \mathrm{Ob}(S)$. Then the comparison arrow*

$$\overline{H_{\phi c}(\lambda_s)}\colon R_{\phi c}(\lim s) \to \lim S_{\phi c}(s)$$

in R_Y is natural in $c \in \mathrm{Ob}(C)$.

For **hom case**, set $\mathcal{X} = \mathrm{CAT}$, $X = C^{\mathrm{op}}$, $Y = Set$, $\phi(c) = C(-, c)$.

Proof idea: Reduce to the 'naturality' in c of the corresponding heteromorphism

$$H_{\phi c}(\lambda_s)\colon R_{\phi c}(\lim s) \Leftrightarrow S_{\phi c}(s).$$

Application: Naturality of limit preservation (2)

Let $H\colon R \nrightarrow S$ a **profunctor**, and $\rho\colon C \to R$ and $\sigma\colon C \to S$ functors.

Definition

A family of heteromorphisms $(\eta_c\colon \rho(c) \Rightarrow \sigma(c) \mid c \in \mathrm{Ob}\, C)$ in H is **natural** if the heteromorphic diagram

$$\begin{array}{ccc} \rho(c) & \xrightarrow{\eta_c} & \sigma(c) \\ {\scriptstyle \rho(a)}\downarrow & & \downarrow{\scriptstyle \sigma(a)} \\ \rho(c') & \xrightarrow[\eta_{c'}]{} & \sigma(c') \end{array}$$

commutes for each arrow $a\colon c \to c'$ in C.

Proposition

Suppose $\lambda_{\sigma(c)}\colon \lim \sigma(c) \Rightarrow \sigma(c)$ *in* H *exists for each* $c \in \mathrm{Ob}\, C$.
A family $(\rho(c) \overset{\eta_c}{\Rightarrow} \sigma(c) \in H \mid c \in \mathrm{Ob}\, C)$ *is natural* ***if and only if*** *the family* $(\rho(c) \xrightarrow{\overline{\eta_c}} \lim \sigma(c) \in R \mid c \in \mathrm{Ob}\, C)$ *of induced arrows is natural.*

Application: Naturality of limit preservation (3)

Lemma

*Let $H\colon R \to S$ be an **indexed** profunctor over a 2-category $\mathcal{X}$. Let C be a category, $X, Y \in \mathcal{X}$ objects and $\phi\colon C \to \mathcal{X}(X, Y)$ a functor. Let $h\colon r \to s$ be a heteromorphism in H_X. Then the heteromorphism*

$$H_{\phi c}(h)\colon R_{\phi c}(r) \to S_{\phi c}(s)$$

in H_Y is natural in $c \in \mathrm{Ob}(C)$.

Proof. We need that the LHS commutes for arrows $a\colon c \to c'$ in C.

$$\begin{array}{ccc} R_{\phi c}(r) & \xrightarrow{H_{\phi c}(h)} & S_{\phi c}(s) \\ {\scriptstyle R_{\phi a}(r)}\downarrow & & \downarrow{\scriptstyle S_{\phi a}(s)} \\ R_{\phi c'}(r) & \xrightarrow[H_{\phi c'}(h)]{} & S_{\phi c'}(s) \end{array} \qquad \begin{array}{ccc} R_{f}(r) & \xrightarrow{H_{f}(h)} & S_{f}(s) \\ {\scriptstyle R_{\theta}(r)}\downarrow & & \downarrow{\scriptstyle S_{\theta}(s)} \\ R_{f'}(r) & \xrightarrow[H_{f'}(h)]{} & S_{f'}(s) \end{array}$$

But it's a special case of RHS, the 2nd axiom's pointwise form. □

Therefore $\overline{H_{\phi c}(\lambda_s)}\colon R_{\phi c}(\lim s) \to \lim S_{\phi c}(s)$ is natural in c. □

Concluding remarks

1. Twofold summary:
   - Identified the 'naturality structure' (indexed profunctor) that is admitted by some prominent examples of universal constructs.
   - Showcased that practical theorems about limits can be proved in this abstract setting.
     - More abstract theorems should make this setting more useful.
2. **Heteromorphic diagrams** make the "interface" of this abstract setting pleasing to work with.
3. Sequel: **Yoneda axioms** (following Street & Walters 1978) on indexed profunctors and more generally on 'diagonal sections' of 2-functors $\mathcal{X}^{\mathrm{coop}} \times \mathcal{X}^{\mathrm{op}} \to \mathrm{C{\scriptstyle AT}}$.
4. Fibrations side: Grothendieck construction over a 2-category and its 'functoriality' with respect to indexed profunctors?
5. Directions for generalisation: • unstrictify, • enrich (esp. along the lines of Shulman's enriched indexed categories), • abstract away from profunctors, • go higher.
6. Details: "Indexed profunctors over 2-categories" in arXiv.

References

Occurrences of indexed profunctors over **1-categories**:

- Wood, R.J. (1982). "Abstract proarrows I". In: *Cahiers de topologie et géométrie diférentielle* 23, no. 3, pp. 279–290.
- Koudenburg, Seerp Roald (2012). "Algebraic weighted colimits". PhD thesis. University of Sheffield. arXiv:1304.4079.
- Shulman, Michael (2013). "Enriched indexed categories". In: *Theory and Applications of Categories* 28, no. 21, pp. 616–695.

Reference regarding 'Yoneda axioms':

- Street, R. and Walters, R. (1978). "Yoneda structures on 2-categories". *Journal of Algebra* 50, no. 2, pp. 350–379.

This work:

- Lee, Sori (2023). "Indexed profunctors over 2-categories". arXiv:2302.06515.