Identity types in predicate logic

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Overview

The **theme** of this work is questions of the form:

Can identity types be added universally

...to a model of type theory without them?

ω-Groupoids?

This question is considered in an extremely truncated version of dependent type theory: many-sorted predicate logic, or indexed preorders.

Summary of findings:

1. The ER-descent construction adds identity objects universally.
2. The PER-descent construction adds partial identity objects universally.
3. Virtualisation promotes partial identity objects to identity objects, universally (but in a sense different from the previous ones).

Identity objects (1)

Let $P = (P^0, P^1)$ be an indexed $(\wedge, \top)$-preorder over a $\times$-category:

- P^0 is a category with binary products,
- P^1 is a functor $(P^0)^{\mathrm{op}} \to \mathrm{Pre}^{\wedge,\top}$.

Definition (in internal logic)

An identity object on an $X \in \mathrm{Ob}(P^0)$ is an $\mathrm{Id}_X \in P^1(X \times X)$, s.t.

1. (*introduction*) $x : X \vdash \mathrm{Id}_X(x, x)$, and
2. (*elimination*) for any $Y \in \mathrm{Ob}(P^0)$ and $p, q \in P^1(X \times X \times Y)$, if

$$x : X, y : Y; p(x, x, y) \vdash q(x, x, y),$$

   then $x : X, x' : X, y : Y; \mathrm{Id}_X(x, x') \wedge p(x, x', y) \vdash q(x, x', y)$.

We say $P \coloneqq (P^0, P^1)$ has identity objects if each X has one.

Identity objects (2)

Definition (properly)

An identity object on an $X \in \mathrm{Ob}(P^0)$ is an $\mathrm{Id}_X \in P^1(X \times X)$, s.t.

1. (*introduction*) $\top \leqslant (X \xrightarrow{\delta} X \times X)^*(\mathrm{Id}_X)$, and
2. (*elimination*) for any $Y \in \mathrm{Ob}(P^0)$ and $p, q \in P^1(X \times X \times Y)$, if

$$(X \times Y \xrightarrow{\delta \times Y} X \times X \times Y)^*(p) \leqslant (X \times Y \xrightarrow{\delta \times Y} X \times X \times Y)^*(q),$$

then $(X \times X \times Y \xrightarrow{\pi_1, \pi_2} X \times X)^*(\mathrm{Id}_X) \wedge p \leqslant q$.

We say $P \coloneqq (P^0, P^1)$ has identity objects if each X has one.

Theorem

An indexed $(\wedge, \top)$-poset over a finite-product category has identity objects if and only if it has 'Lawvere equality', i.e. is an 'elementary doctrine'.

The ER-descent construction

Theorem (Pasquali 2015)

There is a 2-functor $\mathrm{ER}\colon \mathrm{IdxPos}_{\mathrm{sn}}^{\times,1,\wedge,\top} \to \mathrm{ED}$ *that is right 2-adjoint to inclusion.*

Construction (Maietti-Rosolini-Pasquali)

The indexed preorder $\mathrm{ER}(P)$ is given by:

- Objects in $\mathrm{ER}(P)^0$: equivalence relations $(X,\sim)$ in P
- Arrows $(X,\sim_X) \to (Y,\sim_Y)$: $X \xrightarrow{f} Y$ s.t. $\sim_X \leqslant (f\times f)^*(\sim_Y)$
- $\mathrm{ER}(P)^1(X,\sim) \coloneqq \{p \mid \pi_1^*(p) \wedge \sim \leqslant \pi_2^*(p)\} \overset{\text{full}}{\subset} P^1(X)$.

Scholium

The assignment $P \mapsto \mathrm{ER}(P)$ *extends to a 2-functor* $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top} \to \mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top,\mathrm{Id}}$ *that is right biadjoint to the inclusion 2-functor.*

PER-descent construction

Let P be an indexed $\wedge$-preorder over a binary-product category.

Definition

Define $\mathrm{PER}(P)$ in the same way as $\mathrm{ER}(P)$, but with as objects in $\mathrm{PER}(P)^0$ *partial* equivalence relations $(X, \sim)$ in P instead.

Example (Tripos-to-topos construction)

$$\text{Triposes} \xrightarrow[\mathrm{PER}]{(1)} \xrightarrow[\text{`virtualise'}]{(2)} \xrightarrow[\substack{\text{take functional} \\ \text{relations as} \\ \text{arrows}}]{(3)} \xrightarrow[\substack{\text{identify arrows} \\ \text{by extensionality}}]{(4)} \xrightarrow[\substack{\text{take underlying} \\ \text{category}}]{(5)} \text{Toposes}$$

Partial identity objects (1)

Definition (in internal logic)

We say P has partial identity objects if each object $X \in P^0$ is *equipped* with an element $\mathrm{PId}_X \in P^1(X \times X)$, such that

1. (*partial reflexivity*) $\mathrm{PId}_X(x, x') \vdash \mathrm{PId}_X(x, x), \mathrm{PId}_X(x', x')$,
2. (*paravirtual elimination*) for any $Y \in P^0$ and $p, q \in P^1(X \times X \times Y)$, if

$$\mathrm{PId}_X(x, x) \wedge \mathrm{PId}_Y(y, y) \wedge p(x, x, y) \vdash q(x, x, y),$$

   then $\mathrm{PId}_Y(y, y) \wedge \mathrm{PId}_X(x, x') \wedge p(x, x', y) \vdash q(x, x', y)$,

3. each arrow $f\colon X \to Y$ satisfies $\mathrm{PId}_X(x,x') \vdash \mathrm{PId}_Y(f(x),f(x'))$,
4. $\mathrm{PId}_{X \times Y}(x, y, x', y') \dashv\vdash \mathrm{PId}_X(x, x') \wedge \mathrm{PId}_Y(y, y')$.

Remark

Choice of identity objects give partial identity objects.

Partial identity objects (2)

Definition (properly)

We say P has partial identity objects if each object $X \in P^0$ is *equipped* with an element $\mathrm{PId}_X \in P^1(X \times X)$, such that

1. (*partial reflexivity*) $\mathrm{PId}_X \leqslant \langle \pi_1, \pi_1 \rangle^*(\mathrm{PId}_X), \langle \pi_2, \pi_2 \rangle^*(\mathrm{PId}_X)$,
2. (*paravirtual elimination*) for any $Y \in P^0$ and $p, q \in P^1(X \times X \times Y)$, if

$$(X \times Y \overset{\pi_1,\pi_1}{\to} X \times X)^*(\mathrm{PId}_X) \wedge (X \times Y \overset{\pi_2,\pi_2}{\to} Y \times Y)^*(\mathrm{PId}_Y) \wedge$$
$$(X \times Y \overset{\delta \times Y}{\to} X \times X \times Y)^*(p) \leqslant (X \times Y \overset{\delta \times Y}{\to} X \times X \times Y)^*(q),$$

   then $\langle \pi_3, \pi_3 \rangle^*(\mathrm{PId}_Y) \wedge \langle \pi_1, \pi_2 \rangle^*(\mathrm{PId}_X) \wedge p \leqslant q$,
3. each arrow $f \colon X \to Y$ in P^0 satisfies $\mathrm{PId}_X \leqslant (f \times f)^*(\mathrm{PId}_Y)$,
4. $\mathrm{PId}_{X \times Y} \simeq \langle \pi_1, \pi_3 \rangle^*(\mathrm{PId}_X) \wedge \langle \pi_2, \pi_4 \rangle^*(\mathrm{PId}_Y)$.

PER-descent adds partial identity objects universally

Theorem

The assignment $P \mapsto \mathrm{PER}(P)$ extends to a 2-functor $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge} \to \mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\mathrm{PId}}$ that is right biadjoint to the inclusion 2-functor.

Virtualisation (1)

Let P be an *oplaxly sectioned* indexed preorder: each object $X \in P^0$ is equipped with an element $\mathrm{os}_X \in P^1(X)$, and every arrow $f\colon X \to Y$ in P^0 satisfies $\mathrm{os}_X \leqslant f^*(\mathrm{os}_Y)$.

Example

An indexed preorder with partial identity objects is oplaxly sectioned, with $\mathrm{os}_X \coloneqq (X \xrightarrow{\delta} X \times X)^*(\mathrm{PId}_X)$.

Definition

We define $\mathrm{Virt}(P)$ to be the indexed preorder with $\mathrm{Virt}(P)^0 \coloneqq P^0$ and $\mathrm{Virt}(P)^1(X) \coloneqq (\mathrm{U}_{\mathrm{Set}} P^1(X), \overset{\mathrm{v}}{\leqslant})$ where $p \overset{\mathrm{v}}{\leqslant} q$ if and only if $\mathrm{os}_X \wedge p \leqslant q$.

Proposition

$\mathrm{Virt}(P)^1$ *is in fact a Kleisli as well as EM object for a (necessarily idempotent) comonad* $\mathrm{v}\colon P^1 \to P^1$ *in the* Pre*-category* $[(P^0)^{\mathrm{op}}, \mathrm{Pre}^\wedge]_{\mathrm{oplax}}$ *given by* $\mathrm{v}_X(p) \coloneqq \mathrm{os}_X \wedge p$.

(v is the ‘coreader comonad’ associated with the point os of P in $[(P^0)^{\mathrm{op}}, \mathrm{Pre}^\wedge]_{\mathrm{oplax}}$.)

Virtualisation (2)

Remark

1. The os_X become tops in $\mathrm{Virt}(P)$, as $\mathrm{os}_X \wedge p \leqslant \mathrm{os}_X$ for any p.
2. If P has partial identity objects, then the PId_X become identity objects in $\mathrm{Virt}(P)$, for:
   - Id-introduction for PId_X just means $\delta^*(\mathrm{PId}_X)$ is a top.
   - Id-elimination for PId_X in $\mathrm{Virt}(P)$ is equivalent to paravirtual elimination in P, under the other three axioms of partial identity.

Theorem

The assignment $P \mapsto \mathrm{Virt}(P)$ extends to

1. *a 2-functor* $\mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\mathrm{os}} \to \mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\top}$*, as well as*
2. *a 2-functor* $\mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\mathrm{PId}} \to \mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\top,\mathrm{Id}}$

ambidextrously biadjoint to 'the' respective inclusion 2-functor.

The left-biadjoint part also holds with respect to pseudonatural morphisms.

Discussions

1. The ER-descent construction is also a left-biadjoint completion, for adding quotients. (Maietti-Rosolini 2013)

   Under investigation: an analogous result for the PER-descent construction.

2. Directly justify the notion of identity object by embedding (certain) indexed preorders as models of type theory.

3. The first two axioms of partial identity objects as an introduction-elimination pair?

4. Do 1-groupoids complete a model of '1-truncated' type theory with identity types?

   $\vdots$

   Do ∞-groupoids complete a model of type theory with identity types?

More details: for now, an extended abstract on `sorilee.github.io`