Strict bilimits

with an overview of limit notions in 2-categories

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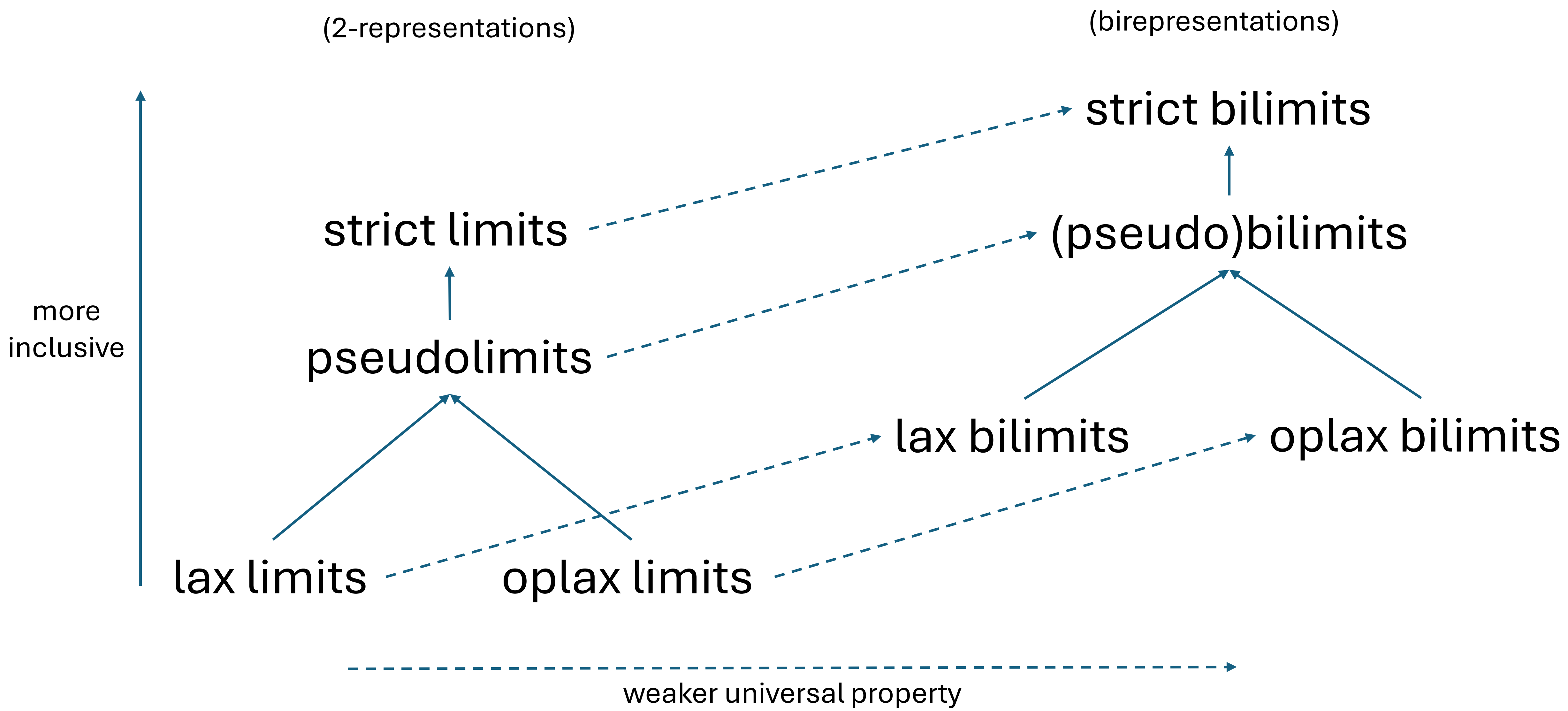
7 Aug 2024

Zulip Category Theory Seminar

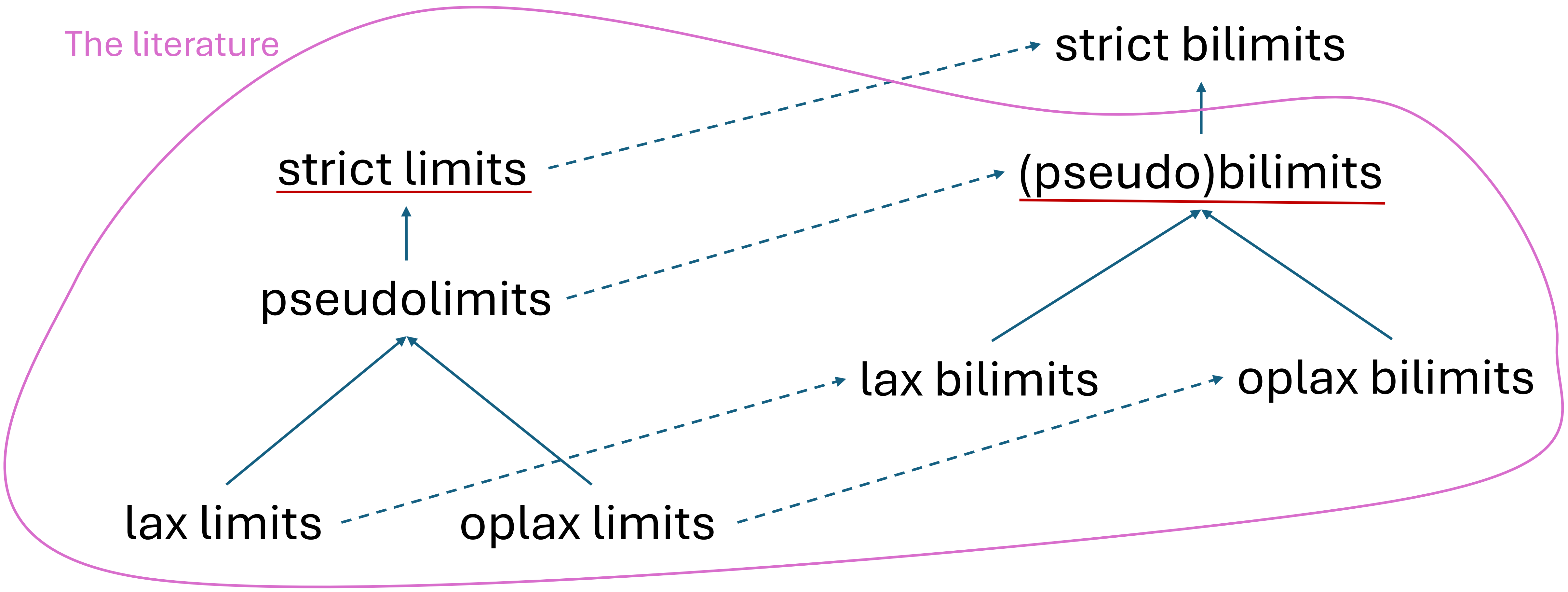
Overview: variants of weighted limits in 2-categories

rather than bicategories!!



The literature is rather silent about strict bilimits, while they are the most general.

Question: are they "unnecessary", or do they have proper examples?



Main observation

Proposition. There are 2-categories A and K, and 2-functors $W: A \to Cat$ and $d: A \to K$, such that

1. d has a W-weighted strict bilimit,
2. d has no W-weighted strict limit, and
3. the weight W is not weakly admitted (see below) by bilimits.

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1. 2-representations vs birepresentations

Definition. Let K be a 2-category. A *2-representation* of a 2-functor $F\colon K \to Cat$ shall refer to a Cat-enriched representation of F, that is, an object $r \in K$ together with an isomorphism

$$\rho\colon K(r, -) \xrightarrow{\cong} F \tag{1}$$

in $[K, Cat]_{\mathrm{s,s}}$.

Definition. Let K be a 2-category. A *birepresentation* of a 2-functor $F\colon K \to Cat$ is an object $r \in K$ together with an equivalence

$$\rho\colon K(r, -) \xrightarrow{\simeq} F \tag{2}$$

in $[K, Cat]_{\mathrm{s,p}}$.

2. Definitions of strict/pseudo/lax/oplax (bi)limit

Let A and K be 2-categories, and let $W\colon A \to Cat$ and $d\colon A \to K$ be 2-functors.

Definition (in words)**.** Let foo = strict, pseudo, lax or oplax.

- A W-weighted foo *limit* of d is a 2-representation for the *Cat*-valued contravariant 2-functor on K of W-weighted foo cones on d.
- A W-weighted foo *bilimit* of d is a birepresentation for the *Cat*-valued contravariant 2-functor on K of W-weighted foo cones on d.

More precisely (strict bilimit):

Definition. A W-weighted *strict bilimit* of d is a birepresentation of the 2-functor

$$K^{\mathrm{op}} \to Cat\colon k \mapsto [A, Cat]_{\mathrm{s,s}}(W, K(k, d-)).$$

3. Strict (bi)limits subsume pseudo, lax and oplax (bi)limits

“Two-dimensional monad theory”

“Flexible limits for 2-categories”

Theorem (special case of Blackwell et al. 1989, Theorem 3.16 for pseudo and lax; Bird et al. 1989, p. 7 for oplax)**.** If A is a small 2-category, then the three inclusion 2-functors

$$[A, Cat]_{\mathrm{s,s}} \hookrightarrow [A, Cat]_{\mathrm{s,p}}, [A, Cat]_{\mathrm{s,l}}, [A, Cat]_{\mathrm{s,o}}$$

have left adjoints $Q_{\mathrm{p}}, Q_{\mathrm{l}}, Q_{\mathrm{o}}$ respectively.

What follows: deduce from this that strict bilimits subsume pseudo, lax and oplax bilimits.

When K is a 2-category, let $[K^{\mathrm{op}}, \mathit{Cat}]_{\mathrm{s,p\&eqv}}$ denote the wide and locally full sub-2-category of $[K^{\mathrm{op}}, \mathit{Cat}]_{\mathrm{s,p}}$ on equivalences.

Corollary. Let A be a small 2-category, K a locally small 2-category, and $W\colon A \to \mathit{Cat}$ and $d\colon A \to K$ 2-functors. Let $\mathrm{foo} \in \{\mathrm{p(seudo)}, \mathrm{l(ax)}, \mathrm{o(plax)}\}$. For each 0-cell $r \in K$, there is an isomorphism of categories[9]

$$[K^{\mathrm{op}}, \mathit{Cat}]_{\mathrm{s,p\&eqv}}(K(-,r), [A, \mathit{Cat}]_{\mathrm{s,s}}\Big(Q_{\mathrm{foo}}(W), \lambda a.\, K(-, da)\Big))$$
$$\cong$$
$$[K^{\mathrm{op}}, \mathit{Cat}]_{\mathrm{s,p\&eqv}}(K(-,r), [A, \mathit{Cat}]_{\mathrm{s,foo}}\Big(W, \lambda a.\, K(-, da)\Big)).$$

Corollary (abridged)**.** There is an isomorphism of categories

$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}(K(-,r), [A, Cat]_{\mathrm{s,s}}\Big(Q_{\mathrm{foo}}(W), \lambda a.\, K(-,da)\Big))$$

$$\cong$$

$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}(K(-,r), [A, Cat]_{\mathrm{s,foo}}\Big(W, \lambda a.\, K(-,da)\Big)).$$

Proof. The left adjoint Q_{foo} gives an isomorphism

$$[A, Cat]_{\mathrm{s,s}}(Q_{\mathrm{foo}}(W), \lambda a.\, K(-,da)) \cong [A, Cat]_{\mathrm{s,foo}}(W, \lambda a.\, K(-,da))$$

in $[K^{\mathrm{op}}, Cat]_{\mathrm{s,s}}$, hence in $[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}$. To this isomorphism, the 2-functor

$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}(\lambda x.\, K(x,r), -) \colon [K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}} \to Cat$$

applies, giving the desired isomorphism[10] in Cat. This proves the corollary.

Corollary (abridged)**.** There is an isomorphism of categories

$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}(K(-,r), [A, Cat]_{\mathrm{s,s}}\Big(Q_{\mathrm{foo}}(W), \lambda a.\, K(-,da)\Big))$$
$$\cong$$
$$[K^{\mathrm{op}}, Cat]_{\mathrm{s,p\&eqv}}(K(-,r), [A, Cat]_{\mathrm{s,foo}}\Big(W, \lambda a.\, K(-,da)\Big)).$$

That is, in simplified words, a W-weighted foo bilimit of d with vertex r is precisely a $Q_{\mathrm{foo}}(W)$-weighted strict bilimit of d with vertex r. This way, strict bilimits subsume pseudo, lax and oplax bilimits.

Remark. We can substitute 'p' with 's' and 'eqv' with 'iso' above, and obtain that that strict limits subsume pseudo, lax and oplax limits.

Remark. Pseudo(bi)limits subsume lax and oplax (bi)limits, by an analogous mechanism (details in the post).

4. A class of strict (bi)limits 'weakly admitting' another

Definition. Let $\mathcal{F}$ and $\mathcal{W}$ be classes of 2-functorial weights, considered as classes of strict limits in 2-categories. We say $\mathcal{F}$ *weakly admits* $\mathcal{W}$ if every 2-category that admits strict limits of type $\mathcal{F}$ admits strict limits of type $\mathcal{W}$.[12] We say $\mathcal{F}$ *(strongly) admits* $\mathcal{W}$ if, in addition to weakly admitting $\mathcal{W}$, every 2-functor that preserves strict limits of type $\mathcal{F}$ preserves strict limits of type $\mathcal{W}$.

Consider now $\mathcal{F}$ and $\mathcal{W}$ as classes of strict bilimits in 2-categories. We say $\mathcal{F}$ *weakly admits* $\mathcal{W}$ if every 2-category that admits strict bilimits of type $\mathcal{F}$ admits strict bilimits of type $\mathcal{W}$. We say $\mathcal{F}$ *(strongly) admits* $\mathcal{W}$ if, in addition to weakly admitting $\mathcal{W}$, every 2-functor that preserves strict bilimits of type $\mathcal{F}$ preserves strict bilimits of type $\mathcal{W}$.

5. (Pseudo)bilimits don't weakly admit strict bilimits

Let $MonCat_{\mathrm{p}}$ denote the 2-category of monoidal categories and strong monoidal functors.

Proposition. $MonCat_{\mathrm{p}}$ does not have strict biequalisers.

Proof. Consider the diagram $\{0\} \underset{1}{\overset{0}{\rightrightarrows}} \{0{,}1\}$ in $MonCat_{\mathrm{p}}$, where $\{0\}$ and $\{0, 1\}$ are regarded as indiscrete monoidal categories (with any choice of a monoidal structure on $\{0, 1\}$). Clearly no monoidal category can be the vertex of a cone on this diagram, because every monoidal category is inhabited. In particular, this diagram has no strict bilimit. This proves the proposition. ∎

Proposition. $MonCat_{\mathrm{p}}$ does not have strict biequalisers. ∎

Since we know $MonCat_{\mathrm{p}}$ is a pseudobilimit-complete 2-category (it is in fact pseudolimit-complete; see Blackwell et al. 1989, Theorem 2.6), it is an example of a pseudobilimit-complete 2-category that does not have strict biequalisers. (In particular, it is an example of a pseudobilimit-complete 2-category that is not strict-limit complete.) Therefore:

Corollary. Pseudobilimits don't weakly admit strict biequalisers. In particular, they don't weakly admit strict bilimits. ∎

6. There is a biequaliser that cannot be given by an equaliser.

We will now prove the 'main observation':

Proposition. There are 2-categories A and K, and 2-functors $W: A \to Cat$ and $d: A \to K$, such that

1. d has a W-weighted strict bilimit,
2. d has no W-weighted strict limit, and
3. the weight W is not weakly admitted (see below) by bilimits.

Construction. Let K be a 2-category. We will define a 2-category K'.

The 0-cells of K' are the 0-cells of K. For each 1-cell $a\colon x \to y$ in K, its two copies $a^0, a^1\colon x \to y$ are 1-cells in K', and all 1-cells in K' are of this form. The 2-cells $f^p \to g^q$ $(p, q \in \{0, 1\})$ in K' are the 2-cells $f \to g$ in K.

The identity 1-cell on a 0-cell $x \in K'$ is the 1-cell id_x^0. If $f^p\colon x \to y$ and $g^q\colon y \to z$ are 1-cells, then their composite is $g^q f^p := (gf)^{\max\{p,q\}}\colon x \to z$. The identity as well as vertical and horizontal composite 2-cells in K' are given by the respective operations in K. This defines K'.[21]

Proposition.

1. K' is a 2-category.
2. The forgetful 2-functor $u\colon K' \to K$ is a biequivalence of 2-categories.
3. Let $W\colon A \to Cat$ be a 2-functor. If K has strict W-(co)limits, then K' has strict W-bi(co)limits.
4. Diagrams of the form $x \overset{f^0}{\underset{g^1}{\rightrightarrows}} y$ admits no strict equaliser in K'.

1. K' is a 2-category.

Proof. 1. The composition of 1-cells is associative, for $\max\{-_1, -_2\}$ is associative. Identity 1-cells are unital, for 0 is unital with respect to $\max\{-_1, -_2\}$. The vertical and horizontal compositions of 2-cells are associative, and identity 2-cells are unital, because the same is the case for the underlying 2-cells in K. For the likewise reason, the horizontal composition of 2-cells preserves identity 2-cells as well as vertical composition. Therefore K' is a 2-category.

2. The forgetful 2-functor $u\colon K' \to K$ is a biequivalence of 2-categories.

The 2-functor $u\colon K' \to K$ is bijective on 0-cells, 1-homwise surjective (splitly) and 2-homwise bijective, hence[22] a biequivalence[23].

3. Let $W\colon A \to Cat$ be a 2-functor. If K has strict W-(co)limits, then K' has strict W-bi(co)limits.

Let $d'\colon A \to K'$ be a 2-functor. In light of 2., we have an equivalence of categories, i.e. an equivalence in the 2-category Cat,

$$K'(x', y') \simeq K(ux', uy')$$

that is strictly natural in $x', y' \in K'_0$.[24] It follows that we have an equivalence

$$K'(x', d'-) \simeq K(ux', ud'-)$$

in the 2-category $[A, Cat]_{\mathrm{s,s}}$ that is strictly natural in $x' \in K'_0$. This induces an equivalence of categories

cones on *d'* in *K'*

$$[A, Cat]_{\mathrm{s,s}}(W, K'(x', d'-)) \simeq [A, Cat]_{\mathrm{s,s}}(W, K(ux', ud'-))$$

cones on *ud'* in *K*

that is strictly natural in $x' \in K'_0$.[25]

3. Let $W\colon A \to Cat$ be a 2-functor. If K has strict W-(co)limits, then K' has strict W-bi(co)limits.

(there is in fact an isomorphism of categories)

Now, let $l \in K_0$ and an equivalence of categories

$$K(x,l) \simeq [A, Cat]_{\mathrm{s,s}}(W, K(x, ud'-))$$

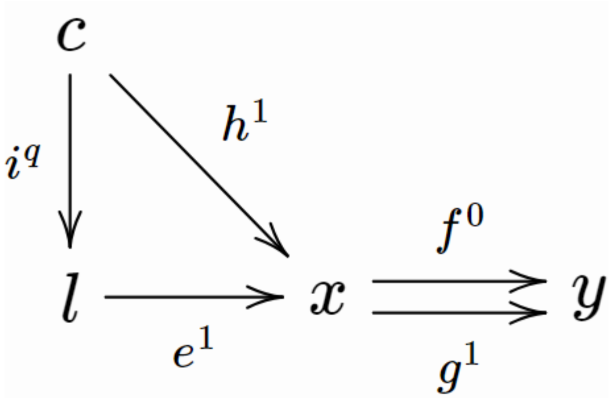
strictly natural in $x \in K_0$ be a strict W-limit of $ud'\colon A \to K$. Let l' be the unique 0-cell in K' such that $l = ul'$. Then we have the chain of equivalences of categories

$$\begin{aligned} K'(x', l') \simeq K(ux', ul') = K(ux', l) &\simeq [A, Cat]_{\mathrm{s,s}}(W, K(ux', ud'-)) \\ &\simeq [A, Cat]_{\mathrm{s,s}}(W, K'(x', d'-)) \end{aligned}$$

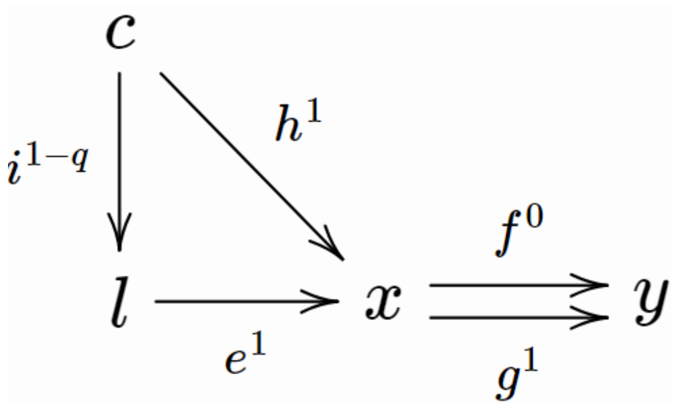
strictly natural in $x' \in K'_0$, providing the 0-cell $l' \in K'$ with the structure of a strict W-bilimit of d'. This proves 3.

4. Diagrams of the form $x \underset{g^1}{\overset{f^0}{\rightrightarrows}} y$ admits no strict equaliser in K'.

If $c \xrightarrow{h^p} x$ is a strict cone on the diagram, then necessarily $p = 1$. Now whenever $c \xrightarrow{h^1} x$ and $l \xrightarrow{e^1} x$ are two strict cones on the diagram and $i^q \colon c \to l$ is a 1-cell such that the triangle in



commutes, then the triangle in



must also commute. Therefore no strict cone on the diagram can satisfy the uniqueness condition of 2-universality. This proves 4.

Corollary.

1. If K is inhabited, then K' does not have strict equalisers.
2. If K is inhabited and has strict equalisers, then K' has strict biequalisers but lacks strict equalisers.
3. If K is strict-limit complete, then K' is strict-bilimit complete but lacks strict equalisers (so is not strict-limit complete).

Proof.

1. As soon as a 0-cell $x \in K'$ exists, the diagram

$$x \underset{\mathrm{id}_x^1}{\overset{\mathrm{id}_x^0}{\rightrightarrows}} x$$

   can be formed, which admits no strict equaliser by the proposition's 4.

2. Immediate by 1. and the proposition's 3.

3. Since K is strict-limit complete, it has a limit of the empty diagram, so is inhabited. Hence also immediate by 1. and the proposition's 3. This proves the corollary.

Proposition. There are 2-categories A and K, and 2-functors $W\colon A \to Cat$ and $d\colon A \to K$, such that

1. d has a W-weighted strict bilimit,
2. d has no W-weighted strict limit, and
3. the weight W is not weakly admitted (see below) by bilimits.

In light of the 2. of the corollary, all we need in order to obtain the counterexample sought above is an inhabited 2-category with strict equalisers. Perhaps the simplest such 2-category is 1, which evidently has all strict limits, in particular strict equalisers. In fact, $1'$ is the promised counterexample by a 2-category with exactly one 0-cell, one non-identity 1-cell and two non-identity 2-cell. 2-categories that induce "non-mini" counterexamples include Cat, which is known to be also strict-limit complete. Thus both $1'$ and Cat' are in fact examples of a 2-category that is strict-bilimit complete but lacks strict equalisers, and are therefore more than sufficient to be our counterexamples.

Question. Is there a "naturally occurring" example of a strict bilimit that is not weakly admissible by pseudobilimits and not equivalent to a strict limit?

- John Bourke told me at CT2024 that Bourke, Lack and Vokřínek (2023), "Adjoint functor theorems for homotopically enriched categories" considers '*E*-weak coequalisers' for *E* the class of surjective equivalences in *Cat*: they are coequalisers whose universal property is given in terms of surjective equivalences of categories, hence should be proper examples of strict bi(co)limits.

Thank you!

The underlying materials and references are available in the post "Strict bilimit and its proper examples" on sorilee.github.io